



Developing Mathematical Thinking:
Multiplication Models, Vocabulary, and Progressions

RESEARCH OVERVIEW

BY

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MULTIPLICATION MODELS, VOCABULARY, AND PROGRESSIONS

(RESEARCH OVERVIEW)

Introduction

Multiplication, a fundamental operation in arithmetic, has a rich history that spans thousands of years, evolving from simple counting methods to complex algorithms. This research overview synthesizes findings from top-tier mathematics education research, including studies by Gravemeijer, Mulligan, Mitchelmore, and others, to explore how learning progressions, interconnected models, precise vocabulary, and historical development enhance students' understanding of multiplication.

Structural Vocabulary

The use of structural vocabulary—such as "iteration" for multiplier, "unit" for multiplicand, and "total" for product—has been shown to improve student understanding of multiplication concepts. These terms provide clarity by emphasizing the underlying structure of multiplication:

1. **Iteration:** Refers to the repeated action inherent in multiplication. Using this term helps students distinguish between multiplication and addition, and reinforces the iterative nature of these operations.
2. **Unit:** Highlights the consistent size of groups being multiplied. This term aligns with visual models, such as arrays or area diagrams, and supports proportional reasoning.
3. **Total:** Reinforces the idea of a product as the result of combining iterations of units.

Language-sensitive interventions that incorporate these terms have been shown to improve conceptual clarity and problem-solving skills (Anghileri, 1989; Mulligan & Mitchelmore, 1997). By explicitly teaching structural vocabulary alongside visual models, educators can help students articulate their reasoning and connect informal strategies to formal operations.

Common Misconceptions

Students often mistakenly believe that multiplication always results in a larger number, which is not true for fractions or negative numbers (Baroody, 1987). Misunderstanding the order of operations can lead to errors in complex calculations, mainly when dealing with parentheses and exponents (Kilpatrick et al., 2001). A common error is simply

adding zeros to the end of a number when multiplying by powers of ten, rather than understanding the increase or decrease in place value (Fosnot & Dolk, 2001).

Learning Progressions in Multiplication

Learning progressions in multiplication are designed to scaffold students' understanding from basic concepts to advanced applications. Early instruction often begins with repeated addition, skip counting, and visual models, such as number lines or bar models. These strategies help students conceptualize multiplication as iterating units or equal groups. Research emphasizes that these foundational methods are critical for transitioning students from additive reasoning to multiplicative thinking (Gravemeijer et al., 2017).

As students advance through elementary school, they encounter more sophisticated models such as area models, ratio tables, and partial products. These representations build on prior knowledge while introducing new concepts, such as the distributive property and place value (Anghileri, 1989; Mulligan & Mitchelmore, 1997). Middle school instruction focuses on formal methods, such as the standard algorithm, while integrating multiplication into broader contexts, including fractions and proportional reasoning (Gravemeijer & Cobb, 2006). Progressive formalization guides this transition by connecting informal strategies to formal representations (Gravemeijer, 1994).

Mathematical Models for Multiplication

Mathematical models provide essential scaffolding for conceptualizing multiplication. Research identifies a sequence of models that support progressive formalization:

1. **Number Lines and Bar Models** introduce multiplication as repeated addition or equal grouping. They help students visualize iterations and units while preparing them for proportional reasoning.
2. **Area Models** represent multiplication geometrically by linking factors to dimensions of a rectangle. This model supports understanding of the distributive property and prepares students for algebraic thinking (Gravemeijer & Doorman, 1999).
3. **Ratio Tables** organize multiplicative relationships systematically, enabling students to identify patterns and reason proportionally (Prediger et al., 2015).
4. **Partial Products** break down multiplication into smaller steps based on place value. This model transitions seamlessly into the standard algorithm while reinforcing conceptual understanding.
5. The **Standard Algorithm** provides a streamlined, procedural approach to multiplication, emphasizing computational efficiency.

Research highlights the importance of connecting these models to one another to foster deeper understanding. For example, linking area models to partial products helps students see how distributive reasoning applies across representations (Gravemeijer et al., 2017). Without these connections, students may struggle to generalize their understanding or apply it flexibly across contexts.

Progressive Formalization

The concept of progressive formalization involves guiding students through a sequence of representations that gradually increase in abstraction while maintaining connections between them (Gravemeijer & Cobb, 2006). For example, students begin with contextual problems using manipulatives. They transition to more formal, iconic representations, such as number lines, bar models, and area models. Then they adopt formal methods such as ratio tables, equations, and algorithms.

Research indicates that explicitly connecting representations fosters both conceptual understanding and procedural fluency in mathematics. By linking different models, students develop schemas—mental structures that integrate knowledge across various contexts, enabling them to retain concepts over time and apply mathematics flexibly in diverse situations (Anghileri, 1989). This interconnected approach ensures a comprehensive understanding of multiplication, enabling learners to transition between informal and formal mathematical operations. Gravemeijer (1999) emphasizes the pivotal role of emergent models in guiding students through this progression, enabling them to build a solid foundation for advanced mathematical thinking.

Patterns

Recognizing patterns and structures in mathematics, particularly in multiplication, from an early age is essential for fostering a deep understanding of mathematical concepts and promoting long-term success. This awareness supports generalization by encouraging children to identify and represent patterns across various mathematical ideas, such as recognizing that multiplying 3×4 involves three iterations of the unit "4," which can also be visualized as an array with three rows of four objects (Mulligan & Mitchelmore, 1997). Such pattern recognition lays the groundwork for early algebraic thinking by helping students see relationships between numbers and operations (Mulligan & Mitchelmore, 1997). By emphasizing patterns and structures in multiplication, educators can provide a strong foundation for advanced problem-solving and algebraic reasoning.

Implications for Teaching

To maximize student learning in multiplication, educators should use a progression of models that build on students' prior knowledge while explicitly connecting representations to deepen understanding. Incorporating structural vocabulary such as "iteration," "unit," and "total" further clarifies concepts and helps students articulate their reasoning. Instruction should be grounded in meaningful contexts to make multiplication relatable before transitioning to abstract methods, ensuring students can connect new knowledge to real-world applications. Additionally, applying principles of progressive formalization guides students from informal strategies, such as using manipulatives, to formal operations, including visual models, ratio tables, and algorithms (Gravemeijer & Cobb, 2006; Prediger et al., 2015). By integrating these approaches into curricula, educators can foster both conceptual understanding and computational fluency, equipping students with the skills needed for advanced mathematical reasoning.

Conclusion

Research on multiplication education underscores the importance of thoughtfully designed learning progressions that connect diverse models through progressive formalization. By integrating structural vocabulary such as "iteration," "unit," and "total," educators can enhance clarity and foster deeper conceptual understanding.

Emphasizing meaningful contexts and recognizing patterns within multiplication further supports students in transitioning from informal strategies to formal operations, equipping them with both procedural fluency and advanced problem-solving skills. Together, these strategies emphasize the enduring importance of multiplication in mathematics education, enabling students to apply their knowledge flexibly across various contexts and establish a solid foundation for future mathematical success.

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Social Media

Research highlights the importance of using visual representations and precise language to foster students' conceptual understanding of multiplication. Tools like number lines, area models, and arrays have proven especially effective for illustrating multiplication as repeated addition, equal groups, or dimensions of a rectangle. These representations help students connect multiplication to real-world contexts while supporting their ability to generalize patterns and reason proportionally.

Moreover, research suggests that structural vocabulary, such as “iteration,” “unit,” and “total,” is crucial for clarifying the underlying concepts of multiplication. For instance, describing 3×4 as “three iterations of the unit ‘4’” reinforces the iterative nature of multiplication and supports flexible reasoning. By combining visual tools with explicit language, educators can help students establish a strong foundation in multiplication, enabling them to transition

seamlessly from informal strategies to formal operations, such as the standard algorithm.

Join us in exploring these impactful strategies to transform how students learn and apply multiplication!



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