



## Developing Mathematical Thinking: The Five Interpretations of Fractions

RESEARCH OVERVIEW

BY

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# THE FIVE INTERPRETATIONS OF FRACTIONS: A RESEARCH OVERVIEW

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(RESEARCH OVERVIEW)

## Introduction

The five interpretations of fractions - part-whole, quotient, measure, operator, and ratio - form a crucial framework for understanding this complex mathematical concept. These interpretations, refined by Thomas Kieren (1980) from his earlier work (Kieren, 1976), provide a comprehensive lens through which students can develop a deep and flexible understanding of fractions.

This approach to fractions goes beyond the traditional part-whole model, offering students multiple perspectives on fractional relationships. By incorporating all five interpretations into instruction, educators can help students develop a more nuanced and versatile understanding of fractions, preparing them for success in advanced mathematics and real-world applications. The following overview explores each of these interpretations in detail, drawing on research to highlight their importance in fraction education.

## Part-Whole Interpretation

Kieren emphasized that the part-whole construct forms the foundation of students' learning about rational numbers (Kieren, 1980). This interpretation relies on the mental actions of decomposing or more specifically partitioning shapes (Simon et al., 2018). From a cognitive psychology perspective, students mentally split or divide a whole unit into equal parts (partitioning) and then consider a subset of those parts (counting) to form the fraction. For example, imagine a square partitioned into four equal units, and 3 are shaded to represent  $\frac{3}{4}$ .

Part-whole interpretation is the most common foundational understanding of fractions, typically introduced at the second-grade level. Teachers often use visual representations such as area models or sets of discrete objects to illustrate part-whole relationships. This model provides a concrete foundation for fraction concepts that young learners easily grasp. However, research suggests that over-reliance on this model can limit understanding of improper fractions, fractions as quantities, and fraction operations (Simon et al., 2018; Lamon, 2020). It is crucial to emphasize that the "whole" in this interpretation can be defined as any quantity, not just one, which helps students conceptualize improper fractions and fractions as measures. Educators should be cautious about overemphasizing the part-whole interpretation, as it may hinder students' ability to conceptualize fractions in other ways, particularly when dealing with fractions as single quantities or in operations.

## Measure Interpretation

The measure interpretation views fractions as points on a number line. This interpretation is crucial for understanding fractions as quantities independent of a "whole" unit (Simon et al., 2018). Cognitively, it requires students to understand fractions as single numbers on a continuous scale rather than just as parts of a whole, representing a significant shift in perspective essential for developing a comprehensive understanding of fractions.

Instructionally, number lines are particularly effective for illustrating this interpretation. Teachers should emphasize how fractions can represent quantities between whole numbers and how different fractions can represent the same quantity depending on the unit of measure (Simon et al., 2018). For example, on a ruler,  $\frac{1}{4}$  inch and  $\frac{2}{8}$  inch represents the same length but uses different unit sizes. Similarly, in cooking,  $\frac{1}{4}$  cup and  $\frac{4}{16}$  cup are equivalent measures. To reinforce this concept, it is important to use precise language: describe the numerator as the count of how many units we have and the denominator as the unit size, which is how many equivalent units are partitioned between 0 and 1. This language helps students visualize fractions as measurements on a number line, reinforcing the idea of fractions as quantities in their own right. For instance,  $\frac{3}{4}$  can be understood as "3 units, where each unit is  $\frac{1}{4}$ ," whether that whole is an inch, a cup, or any other measure.

The measure interpretation, typically introduced in grades 3 through 5, helps students transition from viewing fractions as parts of a whole to understanding them as numbers with specific positions on a number line. This view is essential for later mathematical concepts, particularly in algebra and calculus, and is crucial for comparing fractions and understanding equivalent fractions. Teachers can use bar models as an intermediary step to facilitate this transition. For example, a bar divided into fourths can be reoriented horizontally and labeled with fractions, creating a bridge between the familiar part-whole concept and the more abstract number line representation.

This approach helps students see how the parts in a part-whole model correspond to intervals on a number line, supporting their understanding of fractions as quantities. For instance, locating  $\frac{5}{4}$  on a number line requires students to understand that the space between 0 and 1 and all subsequent whole numbers is partitioned into 4 equivalent smaller units. To reach  $\frac{5}{4}$ , they need to count 5 of these  $\frac{1}{4}$  units, moving beyond the unit of 1. This approach reinforces the idea of fractions as quantities that can be less than, equal to, or greater than 1 and helps students visualize improper fractions. Despite the initial challenge some students may face in shifting from the concrete part-whole model to this representation, the measure interpretation remains vital in developing a robust understanding of fractions and their relationships to other numbers.

## Quotient Interpretation

Kieren (1980) identified the quotient interpretation as a distinct sub-construct. This view of fractions as division is typically introduced in late elementary or early middle school. Lamon (2020) emphasizes the importance of this interpretation in bridging students' understanding of whole number operations to fractional concepts.

Cognitively, this interpretation requires students to understand the relationship between division and fractions, a concept that builds on their understanding of whole-number operations (Simon et al., 2018). Teachers can use equal sharing scenarios to illustrate this concept, such as dividing food among friends.

The quotient interpretation bridges students' knowledge of division and fractions, laying the groundwork for fractional operations. However, the dual nature of fraction notation—representing both a division process and its result—can confuse learners. For example, consider the fraction  $\frac{3}{4}$ . This can be interpreted in two ways. As a *process*, it represents dividing 3 by 4. As a *result*, it represents the quantity that results from this division, which is 0.75. to be clearer. Example: As a process, the fraction  $\frac{3}{4}$  represents the operation of dividing 3 by 4. As a result, the fraction represents the quotient obtained from this division, which is 0.75.

## Ratio Interpretation

Another of Kieren's (1980) sub-constructs is the ratio interpretation, which compares two or more quantities. Lamon (1999) has extensively researched this interpretation, highlighting its crucial role in developing proportional reasoning skills. Cognitively, this requires students to understand fractions as relationships, which can be challenging for some students (Simon et al., 2018).

Instructionally, teachers can use real-world examples involving comparisons, such as the ratio of boys to girls in a class. This interpretation is crucial for developing proportional reasoning skills (Simon et al., 2018). For example, in a class of 20 students with 12 girls and 8 boys, the ratio of girls to boys can be expressed as  $\frac{12}{8}$ , which simplifies to 3:2. As a fraction, this is  $\frac{12}{8}$  or  $\frac{3}{2}$ , meaning for every 3 girls, there are 2 boys. Alternatively, this can be interpreted as girls representing 150% of the boys in the class. Conversely, the ratio of boys to girls can be expressed as 8:12, which simplifies to 2:3. As a fraction, this is  $\frac{8}{12}$  or  $\frac{2}{3}$ , meaning for every 2 boys, there are 3 girls. This can also be interpreted as boys representing approximately ~67% of the girls in the class. This example demonstrates how ratios can be expressed in multiple ways and how they relate to percentages, enhancing students' understanding of proportional relationships.

## Operator Interpretation

Kieren (1980) included the operator interpretation in his framework, viewing fractions as transformations applied to quantities. Behr et al. (1993) further elaborated on this concept, describing it as a "stretcher/shrinker" or "duplicator/partition" action. Cognitively, this interpretation requires students to understand fractions as functions that can change other values (Simon et al., 2018).

Instructionally, teachers can use examples of scaling or resizing to illustrate this concept, such as finding half or a third of a number (Simon et al., 2018). Lamon (2020) emphasizes that effective fraction instruction should help students understand fractions as numbers in their own right, capable of being manipulated with arithmetic.

In fifth grade, the operator interpretation of fractions allows us to understand fractions as transformations applied to quantities. Here are two examples demonstrating this concept: increasing the size of a picture. Let us say we have a picture with an original size of 20 units. To increase its size by  $\frac{3}{2}$  (or 150%), we multiply the original size by  $\frac{3}{2}$ :  $20 * (\frac{3}{2}) = 30$ . The new size of the picture is 30 units, which is 1.5 times larger than the original. Now, let us decrease the size of a picture. Take the same picture with an original size of 20 units and decrease its size by  $\frac{1}{2}$  (or 50%):  $20 * (\frac{1}{2}) = 10$ . The new size of the picture is 10 units, which is about one-half of the original size.

These examples illustrate how fractions can act as operators to transform quantities, either increasing or decreasing them. This interpretation helps students understand fractions as dynamic entities that can modify other values, laying a foundation for more advanced mathematical concepts.

## Conclusion

In conclusion, Kieren and Lamon's work has significantly shaped our understanding of fraction concepts and instruction. Their research underscores the importance of exposing students to all five interpretations to develop a comprehensive understanding of fractions over time, providing a robust foundation for advanced mathematical concepts. A balanced approach to instruction, incorporating all five interpretations, can help students develop a flexible and deep understanding of fractions.

Effective teaching should use a variety of representations and real-world contexts to help students grasp these different interpretations and their interconnections (Simon et al., 2018). By addressing both the strengths and limitations of each interpretation, educators can guide students toward a more comprehensive and nuanced understanding of fractions, preparing them for success in more advanced mathematical studies.

## References

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## Social Media

Is Your Fraction Foundation Solid? Test Your Knowledge of the 5 Key Interpretations of Fractions. They are part-whole, measure, quotient, ratio, and operator. These interpretations have emerged as a robust framework that serves educators in the critical role of developing students' conceptual understanding of fractions over time. Moreover, the research suggests a shift in instructional approach, moving beyond traditional part-whole descriptions to emphasize all five interpretations. This framework fosters a more nuanced comprehension of fractional situations. By implementing these evidence-based strategies, educators can cultivate flexible thinking about fractions, thereby equipping students for success in advanced mathematics and real-world applications.

Join us in exploring these powerful learning strategies and their impact on mathematical thinking!

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