



Developing Mathematical Thinking:
How Varied Practice Transforms Math Learning

DMTI INSIGHTS

BY

Jonathan Brendefur, PhD
and Associates

FROM ROUTINE TO REASONING: HOW VARIED PRACTICE TRANSFORMS MATH LEARNING

(DMTI INSIGHTS)

Introduction

What if practice could build mathematical flexibility instead of rigid routine?

In many classrooms, practice means pages of nearly identical problems—20 addition facts, 15 fraction conversions, or a full sheet of long division. Students learn to repeat, but not to reason. They may master procedures, yet fail to connect them to meaningful contexts or visual representations. This separation between **contextual understanding**, **visual modeling**, and **symbolic notation** weakens transfer and limits flexibility.

Varied Practice offers a different approach. Each task invites students to work across **three representations of mathematical ideas**—the **contextual** (story or situation), the **iconic** (visual model), and the **symbolic** (equation, algorithm, or verbal explanation). In a **three-column format**, one column is provided while the other two must be constructed. For instance, a student might be given a bar model and must write both the matching story problem and symbolic equation. This intentional translation among forms develops deep understanding and flexible reasoning.

This structure is grounded in decades of research from **mathematics education, cognitive psychology, and learning science**. It draws from Jerome Bruner's modes of representation, Allan Paivio's dual coding theory, and the mathematics education literature on multiple representations and structural reasoning. Together, these frameworks explain why **moving among story, model, and symbol** is not just pedagogically sound—it's cognitively powerful.

Theoretic Foundations

How do students build mathematical meaning?

As Jerome Bruner (1966) argued, students construct understanding through three interconnected modes of representation: **enactive** (action-based), **iconic** (visual or pictorial), and **symbolic** (abstract or linguistic). In mathematics, these correspond naturally to manipulatives and real-world actions (enactive), visual models such as bar models or number lines (iconic), and equations or algorithms (symbolic). When instruction emphasizes all three modes—and the ability to move between them—students build **representational fluency**, the capacity to express a single idea in multiple forms and recognize their underlying equivalence.

Modern mathematics education researchers (Ainsworth, 2006; Lesh, Post, & Behr, 1987) affirm that **multiple representations** are essential for deep conceptual understanding. Ainsworth describes the “complementary roles” of different representations: visuals reveal relationships that symbols conceal, while symbols allow for generalization and abstraction beyond a specific model. When students can flexibly translate across these modes, **they are better prepared to apply mathematics to new situations.**

For teachers, this framework reframes practice itself: not as repetition of form, but as variation of representation. A first grader using a bar model showing 14 red apples and 9 green apples to match $14 + 9 = 23$, or a fifth grader linking a fraction bar model to the equation $\frac{3}{4} \times 24 = 18$, are each engaging in **meaning-making across Bruner’s modes.**

Dual Coding Theory: Why Seeing and Saying Math Strengthens Memory

*Why is the **iconic model** so powerful?*

Cognitive psychologist Allan Paivio’s **Dual Coding Theory** (1971, 1986) provides the neurological explanation for why varied practice works so effectively. Paivio proposed that the human mind encodes information through **two interconnected systems**: a **verbal channel** for language and a **non-verbal channel** for imagery. When ideas are encoded through both, recall and understanding improve dramatically because learners build **two pathways** to access the same knowledge.

Engaging both channels simultaneously:

- Activates more regions of the brain, forming stronger neural connections (Clark & Paivio, 1991).
- Provides multiple retrieval cues (verbal and visual), strengthening long-term memory.
- Reduces cognitive load by distributing processing across both channels, aiding comprehension (Sweller, 1994).

In mathematics, story problems primarily activate the **verbal** system, while visual models engage the **non-verbal** system. **Equations and algorithms**—though symbolic—occupy a middle ground within the verbal channel, functioning as a *language of structure* (Sfard, 2008). When equations are intentionally paired with visual models, such as linking a bar model to $3 + 2 = 5$, both channels operate in tandem. This **dual-coded representation** allows students to move fluidly between seeing relationships and expressing them symbolically.

Students who engage in dual coding are not merely memorizing equations—they are constructing **mental images of structure**. For example, when solving $24 \div 6 = 4$, a student who pictures “24 fish divided evenly among 6 trays” is leveraging both cognitive systems. The equation becomes meaningful because it is anchored in imagery and context.

Cognitive Benefits: Translating Representations Builds Transferable Understanding

How does this design change the way students learn?

The act of **translating between representations** is cognitively demanding and deeply generative. It forces students to reorganize and recontextualize their knowledge, building flexible, interconnected schemas rather than isolated facts.

Research across cognitive psychology and mathematics education identifies several key benefits:

Conceptual Understanding Beyond Procedures

Creating or interpreting a representation demonstrates what a student truly understands. For example, when given the equation $9 - 5 = 4$, a child who writes “I had 9 apples and gave away 5” shows a *separating* model of subtraction. A child who writes “Tom has 9 apples and Mary has 5; Tom has 4 more” shows a *compare* model—revealing a deeper grasp of subtraction’s multiple structures.

Metacognition and Self-Monitoring

Translating between forms naturally encourages reflection. When a student’s bar model does not align with their equation, a cognitive conflict arises—prompting self-correction and metacognition (Schoenfeld, 2016). Varied practice, by design, builds in this feedback loop.

Mathematical Flexibility and Transfer

Real-world problems rarely appear in symbolic form. To solve them, students must move from situation → iconic model → symbols, and often back again. Practicing these translations develops **cognitive flexibility**, enabling transfer to novel tasks (Rittle-Johnson & Star, 2007). Over time, students internalize not only procedures, but the **relationships** among representations.

In short, **mathematical meaning lives in the movement**—the active coordination between story, image, and symbol. When students can move flexibly among them, their knowledge becomes transferable, durable, and richly interconnected.

Designing Varied Practice: Making Representations the Practice

How can teachers bring this idea to life?

Effective varied practice is not accidental—it’s intentionally designed to challenge students to connect representations. A well-constructed **Three-Column Worksheet** (DMTI, 2020) can serve as the vehicle for this practice. Each row addresses a single mathematical idea and includes three columns:

(1) Contextual (Story) | (2) Iconic (Visual Model) | (3) Symbolic / Language (Equation or Explanation)

Only one column is filled in; students must generate the others. For example:

- A teacher gives an equation ($8 + 4 = ?$; $8 + 4 = 12$). Students create a bar model or number line and write a story.
- Another time, students are given a bar model and must create the corresponding story and equation.

Design principles include:

- **Consistency:** Ensure the same mathematical relationship underlies all three columns.
- **Variation:** Alternate which column is given, avoiding predictable patterns.

- **Progression:** Begin with two provided columns (to scaffold) and move toward generating two from one.
- **Intentional models:** Use bar models, number lines, or area models that naturally represent the concept.

Optional extension: The third column may also focus on **language**, where students describe the conceptual action (“I partitioned one into 4 equal units”). This encourages precision in mathematical communication and deepens conceptual awareness.

When used regularly, these tasks transform worksheets into cognitive workouts—promoting not only skill fluency but conceptual agility. They make visible what students know and how they think.

Conclusion: Building Thinkers, Not Just Solvers

Varied Practice is more than a worksheet strategy—it is a cognitive framework for **building relational understanding**. It integrates the insights of **Bruner’s representational theory**, **Paivio’s dual coding**, and **modern structural mathematics education** into a single, powerful classroom routine. By requiring students to move between story, iconic model, and symbol, we teach them to see mathematics not as disconnected tasks, but as a coherent system of meaning.

For educators, the essential question shifts from “*Can my students compute the answer?*” to “*Can my students show the meaning in multiple ways?*” This approach transforms practice from repetition to reasoning, helping students become thinkers who understand the *why* behind the *how*.

As the Developing Mathematical Thinking Institute emphasizes, **mathematical structure is the bridge between context, iconic model, and symbol**. When students compose, decompose, iterate, and partition across representations, they are not just doing math—they are developing the habits of thought that make mathematical reasoning a lifelong tool.

References

- Ainsworth, S. (2006). *DeFT: A conceptual framework for learning with multiple representations*. *Learning and Instruction*, 16(3), 183–198.
- Bruner, J. (1966). *Toward a theory of instruction*. Harvard University Press.
- Clark, J. M., & Paivio, A. (1991). Dual coding theory and education. *Educational Psychology Review*, 3(3), 149–210.
- Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Lawrence Erlbaum.
- Paivio, A. (1971). *Imagery and verbal processes*. Holt, Rinehart & Winston.
- Paivio, A. (1986). *Mental representations: A dual coding approach*. Oxford University Press.
- Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural

knowledge? *Journal of Educational Psychology*, 99(3), 561–574.

Schoenfeld, A. H. (2016). *How we think: A theory of goal-oriented decision making and its educational applications*. Routledge.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, 4(4), 295–312.

& Taylor, K. (2007). The shuffling of mathematics problems improves learning. *Instructional Science*, 35(6), 481–498.

Social Media

Are We Teaching Math Procedures or Building Mathematical Thinkers?

It is easy to create worksheets where every problem looks the same. Students learn to mimic a procedure and get the right answer. However, does this build a true, flexible understanding? Research says no. If we want students who can solve real-world problems, we need to change how we practice.

Our new DMT Insights dives into **Varied Practice Worksheets**—a simple but powerful shift:

- **Why** moving between stories, iconic models, and equations builds deeper neural pathways than repetition alone.
- **How** a simple 3-column format (Story - Visual - Equation) forces students to *think*, not just compute.
- **The key cognitive science** (like Dual Coding Theory) that makes this approach so effective for long-term learning.
- **Practical ways** to scaffold this in any elementary classroom, for any topic.

Ready to move beyond answer-getting and toward building agile mathematical minds?

What is one topic (like fractions or word problems) where your students struggle to connect the idea to the procedure? Share below!



For more information, contact@dmtinstitute.com or follow us on social media:



facebook.com/dmtinstitute



instagram.com/dmtinstitute/



[@DMTInstitute.bsky.social](https://DMTInstitute.bsky.social)



mathsuccess.dmtinstitute.com



x.com/dmtinstitute



linkedin.com/company/dmt-institute



Tiktok.com/@math.success



youtube.com/@dmtinstitute

