

# DMT INSIGHTS

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## FRACTIONS: THE DEVELOPMENTAL UNDERSTANDINGS BENEATH CONCEPTS AND OPERATIONS

### Introduction

Few topics in the K–8 curriculum carry as much long-term weight, or cause as much difficulty, as fractions. Fractions are the first sustained encounter students have with numbers that do not behave like the counting numbers, and they mark the point at which many students' mathematical confidence begins to erode. Yet the stakes could hardly be higher. Fraction knowledge in elementary school is one of the strongest predictors of algebra achievement and overall mathematics attainment years later, even after accounting for whole-number skill, IQ, working memory, and family background (Siegler et al., 2012; Bailey, Hoard, Nugent, & Geary, 2012). The National Mathematics Advisory Panel (2008) concluded that difficulty with fractions is pervasive and constitutes a major obstacle to further progress in mathematics, including algebra.

From a mathematics education perspective, fractions are not a single idea but a family of related meanings—part-whole, measure, quotient, operator, and ratio—that students must gradually coordinate (Kieren, 1976; Behr, Lesh, Post, & Silver, 1983). From a cognitive perspective, learning fractions requires a substantial reorganization of what a number is: students must extend their concept of number beyond counting to include quantities that can be infinitely partitioned and ordered by magnitude (Siegler, Thompson, & Schneider, 2011). Understanding both of these demands—the mathematical structure and the cognitive reorganization—is essential for learning and teaching fractions well. Simon (2006) termed these reorganizations key developmental understandings—conceptual advances that make later fraction concepts and procedures learnable rather than merely memorable.

**Key Idea • Fraction knowledge is among the strongest predictors of later algebra and mathematics achievement, yet fractions demand a reorganization of what a number is, which is exactly why they are both so consequential and so difficult.**

### What Is a Fraction? The Subconstructs of Rational Number

A central insight from decades of research is that the symbol  $\frac{3}{4}$  does not have a single meaning. Building on Kieren's (1976) foundational analysis, Behr, Lesh, Post, and Silver (1983) described rational number as comprising several interrelated subconstructs: part-whole (3 of 4 equal parts of a whole), measure (a distance of  $\frac{3}{4}$  along a number line, or  $\frac{3}{4}$  as an iteration of the unit  $\frac{1}{4}$ ), quotient (3 divided by 4, the result of sharing 3 items among 4 people), operator (a

function that scales a quantity, as in  $\frac{3}{4}$  of 12), and ratio (a multiplicative comparison of two quantities, 3 to 4). Competent understanding requires students to recognize these meanings, distinguish among them, and move flexibly among them.

Instruction in the United States has traditionally overemphasized the part-whole subconstruct—typically shaded regions or “pieces of a pizza”—at the expense of the others (Charalambous & Pitta-Pantazi, 2007). While part-whole provides an accessible entry point, it is a limited foundation. It does not readily explain improper fractions (how does one shade 7 of 4 parts?), it obscures the idea that a fraction is itself a number with a location and a magnitude, and it does not connect naturally to division or measurement. Empirical work confirms that the part-whole and measure subconstructs play a particularly important role, and that the measure subconstruct in particular supports operations and equivalence (Charalambous & Pitta-Pantazi, 2007). A robust fraction curriculum therefore develops all of the subconstructs deliberately, with special attention to fractions as measured magnitudes.

**Key Idea • A fraction is not one idea but several, part-whole, measure, quotient, operator, and ratio, and instruction should develop all of them, with particular attention to fractions as measured magnitudes rather than only shaded parts.**

## The Whole Number Bias: Reorganizing the Concept of Number

Perhaps the most documented obstacle in fraction learning is whole number bias: students' tendency to apply the properties of whole numbers inappropriately to fractions (Ni & Zhou, 2005). Because students spend their early years reasoning exclusively about counting numbers, they build strong, well-practiced intuitions—larger numerals mean larger quantities, every number has a unique next number, multiplication makes bigger, and division makes smaller. Each of these intuitions fails for fractions.

Every upper-elementary teacher knows the consequences. Students judge that  $\frac{1}{8}$  is greater than  $\frac{1}{3}$  “because 8 is larger than 3.” They add  $\frac{1}{2} + \frac{1}{3}$  and get  $\frac{2}{5}$  by operating on numerators and denominators separately, as though each were an independent whole number. They are surprised that multiplying by  $\frac{3}{4}$  yields a smaller result, or that there are infinitely many numbers between 0 and 1. The language of fractions can reinforce the bias: naming  $\frac{1}{3}$  as “one out of three” frames it as a count of parts and obscures that it is a single number, whereas naming it “one-third,” like “one-half,” treats the fraction as one quantity (Paik & Mix, 2003). Ni and Zhou (2005) argue that whole-number bias has both a natural cognitive origin—count-based number sense emerges early and is heavily reinforced—and an instructional origin, in curricula that introduce fraction procedures before students have reconceived what these new numbers are. From this perspective, the aim of assessment and instruction together is not merely to add new rules but to help students reorganize their concept of number so that whole numbers and fractions are understood as points on a common continuum; a diagnostic that surfaces where that reorganization has stalled makes the learning teachable.

**Key Idea • The whole number bias, importing whole-number rules into fractions, is the central obstacle in fraction learning; the aim of diagnosis and instruction together is not more rules but a reorganization of number so that whole numbers and fractions belong to a common continuum.**

### CLASSROOM SNAPSHOT

**Teacher:** “Which is larger,  $\frac{1}{3}$  or  $\frac{1}{4}$ ?”

**Student A:** “ $\frac{1}{4}$ , because 4 is larger than 3.”

**Student B:** “ $\frac{1}{3}$ . If you cut one into 3 equal parts, each part is larger than if you cut it into 4.”

**Teacher:** “How could we settle it?”

**Student B:** “Put both on a number line from 0 to 1.  $\frac{1}{3}$  is farther from 0.”

Both students can compute; they differ in whether they reason about the size of the unit fraction or the size of the denominator—the center of the whole-number bias.

## Magnitude and the Number Line: An Integrated Theory of Numerical Development

The most influential recent framework here is Siegler's integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011; Siegler, Fazio, Bailey, & Zhou, 2013). Its central claim is simple: what unifies whole numbers, fractions, decimals, and negatives is that they all have magnitudes that can be located on a number line. Numerical development, on this account, is the process of progressively broadening the set of numbers whose magnitudes a learner can represent and order, and of coming to understand that all real numbers share this property.

This theory reframes fractions as the decisive test case for theories of numerical development, because fractions are precisely where the count-based conception of number breaks down and a magnitude-based conception must take over (Siegler et al., 2013). Empirically, the accuracy with which students place fractions on a number line is strongly associated with overall fraction knowledge and with broader mathematics achievement, often more strongly than part-whole or procedural measures (Siegler et al., 2011). The number line is powerful precisely because it represents the measure subconstruct directly: it shows that  $\frac{3}{4}$  is a single number with a definite size, that fractions can be compared as lengths, and that whole numbers and fractions live on the same continuum. For testing and instruction alike, this implies that the number line should be a central, not incidental, representation in fraction learning. Benchmarks of 0,  $\frac{1}{2}$  and 1 serve as anchors for this reasoning: a student who knows that  $\frac{3}{4}$  lies nearer 1 than  $\frac{1}{2}$ , or that  $\frac{1}{8}$  lies nearer 0, has a rough sense of size that constrains finer comparisons and flags an impossible result such as  $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ . Consistent with this, in a longitudinal study of 357 children, number-line estimation emerged as the strongest independent predictor of later fraction concepts and procedures among the cognitive and mathematical variables examined (Jordan et al., 2013).

**Key Idea •** What unifies whole numbers and fractions is that both have magnitudes locatable on a number line; number-line placement is among the strongest predictors of fraction knowledge, which is why the number line belongs at the center of learning and instruction.

## Developmental Foundations: Equal Sharing, Partitioning, and Unitizing

The number line makes a fraction's magnitude visible, but students can use it meaningfully only after they build the understandings beneath it, especially partitioning and unitizing. Long before formal fraction symbols appear, young

children possess intuitive resources for reasoning about fair shares. Research in the tradition of Cognitively Guided Instruction shows that children as young as first grade can solve equal-sharing problems, dividing, say, several apples among a group of people, and that these informal partitioning strategies form a productive foundation for fraction concepts (Empson, 1999). In Empson's (1999) study, sustained work with equal-sharing tasks helped young students develop meaning for fractional quantities and for the crucial idea that the units must be equal.

Two developmental ideas are central. The first is partitioning: dividing a quantity into equal units and coordinating the number of units with their size, so that students come to see that more units (parts) means smaller units (parts). The second is unitizing: understanding what counts as “one” and recognizing that the unit can be flexibly redefined—that  $\frac{3}{4}$  can be seen as three iterations of the unit  $\frac{1}{4}$ , or that a group of objects can itself be treated as a single unit (Behr et al., 1983; Lamon, 2007). Analyses of children's fraction schemes describe how learners construct increasingly sophisticated conceptions of the unit and of iteration over time, and how these schemes underpin later reasoning about equivalence and operations (Steffe & Olive, 2010). Realistic Mathematics Education similarly emphasizes building fractions from meaningful contexts of fair sharing and measurement before formalizing them (Streefland, 1991). Learning-trajectory research in the measurement tradition places this unit-and-iteration relationship at the very center of early fraction development, treating fraction-as-measure, rather than shaded parts of objects, as a foundational understanding (Simon, 2006; Simon, Placa, Avitzur, & Kara, 2018). The instructional implication is that fraction understanding should be built from partitioning and sharing activity, not introduced as ready-made symbolic rules.

**Key Idea • Fractions should be built from children's own partitioning and equal-sharing, establishing the unit, then iterating it, rather than introduced as ready-made symbolic rules.**

## Cognitive Foundations: Conceptual and Procedural Knowledge

Why is fraction arithmetic, in particular, so persistently difficult? Lortie-Forgues, Tian, and Siegler (2015) offer a detailed analysis pointing to both inherent and culturally contingent causes. Some difficulty is inherent: fraction operations violate whole-number expectations (multiplication can make smaller), the algorithms are notational and non-transparent (invert-and-multiply has no obvious meaning), and different operations require different treatments of the denominator. Other difficulties are contingent on how fractions are taught, including limited emphasis on magnitude and heavy reliance on memorized procedures.

This connects to a broader theme in mathematics learning: the relationship between conceptual knowledge (understanding of principles and relationships) and procedural knowledge (knowledge of steps and algorithms). Both matter, and in fractions they appear to develop in a mutually reinforcing, bidirectional relationship, with conceptual understanding of magnitude supporting the meaningful use and retention of procedures (Hecht & Vagi, 2010). When procedures are learned in isolation from magnitude—when  $\frac{1}{2} + \frac{1}{3}$  is a symbol-manipulation task disconnected from any sense of size—students have no way to detect that an answer of  $\frac{2}{5}$  is impossible, because it is smaller than one of the addends. Similarly, when asked to estimate fraction sums, a large share of students in Grades 4–8 produce answers smaller than one of the addends being combined—direct evidence that fraction procedures can develop without magnitude-based constraints (Braithwaite, Tian, & Siegler, 2018). Structured, magnitude-grounded representations reduce this kind of error by giving students a conceptual check on procedural work, echoing the well-established finding

that learning organized around underlying structure transfers more readily than learning organized around surface features (Ni & Zhou, 2005; Lortie-Forgues et al., 2015).

**Key Idea • Conceptual and procedural knowledge grow together; when procedures are learned apart from magnitude, students cannot tell an impossible answer from a correct one, which is why magnitude must anchor computation.**

## Building the Fraction Math Learning Profile: What the Diagnostic Should Measure

The research reviewed here leads to a practical question: if understanding fractions is this consequential and this fragile, what should a diagnostic actually measure? A Math Learning Profile (MLP) for fractions is not a score on a page of computation items; it is a map of the developmental understandings a student has, and has not yet, constructed, built so that a teacher can see where a student's reasoning begins to break down. The literature points to a small set of constructs that predict later fraction learning and that together define what such a profile should assess.

Five understandings are primary, because they are the ideas that fraction concepts and operations rest upon. The first is the **unit of one**: whether a student recognizes what has been defined as the unit of one, understands that every fraction is named in relation to it, and can reconstruct the unit of one when only a fractional part is shown, for example, rebuilding one from three-fourths (Lamon, 2007). The second is **partitioning**: whether a student can partition the unit of one into equal-sized fractional units and reason that the number of equal partitions determines the size of each unit, so that more partitions produce smaller units (Lamon, 2007). The third is **iterating**: whether a student understands  $\frac{1}{b}$  as a fractional unit made by partitioning and can iterate it to build  $\frac{a}{b}$ , coordinating the denominator as the unit size and the numerator as the count, including quantities greater than one (Behr, Lesh, Post, & Silver, 1983). The fourth, and the most predictive, is **fraction magnitude**: whether a student treats a fraction as a single number with a size, can place it on a number line, and can compare it against benchmarks such as 0,  $\frac{1}{2}$ , and 1 (Siegler, Thompson, & Schneider, 2011; Jordan et al., 2013). The fifth is **equivalence**: whether a student recognizes the same magnitude expressed with different-sized fractional units, coordinating a change in unit size with the corresponding change in the number of units, rather than as the output of a rule applied to numerator and denominator (Charalambous & Pitta-Pantazi, 2007). Equivalence coordinates the change in unit size with the change in the number of units, so that as fourths become eighths the three units become six, and it is reversible: a student can partition back into larger units to recover  $\frac{3}{4}$  from  $\frac{6}{8}$  (Behr, Lesh, Post, & Silver, 1983).

Two further understandings are drawn from neighboring constructs and assumed rather than taught here: the quotient meaning of a fraction, which grows out of equal-sharing and belongs to work on division (Empson, 1999), and the operator and ratio meanings, which draw on multiplicative comparison and belong to Scaling and Comparing. A diagnostic built on these constructs should probe each in more than one representation, region, length, and number line, and should surface the whole number bias directly, because a student who places  $\frac{1}{8}$  beyond  $\frac{1}{3}$  has revealed a specific, addressable misconception rather than a wrong answer. Read across the developmental trajectory, these constructs locate not merely whether a student can do fractions today, but which understanding, if strengthened, would change what the student can do next.

**Key Idea • A fraction Math Learning Profile should be built on the developmental understandings that predict later learning, the unit of one, partitioning, iterating, fraction magnitude, and equivalence, each probed across representations, so that the diagnostic identifies which understanding, not merely which answer, a student is still constructing.**

## Conclusion

Fractions occupy a pivotal place in the K–8 curriculum because they demand something new: a reorganization of the concept of number to include quantities that are partitioned, ordered by magnitude, and endowed with multiple interrelated meanings. From a mathematics education perspective, helping students learn fractions well means developing the full family of subconstructs—part-whole, measure, quotient, operator, and ratio—with particular emphasis on fractions as measured magnitudes. From a cognitive perspective, it means helping students overcome whole-number bias and locate fractions on a shared number line, so conceptual understanding of magnitude can anchor and give meaning to procedures.

Despite decades of research, fraction teaching in many classrooms remains heavily procedural and centered on part-whole, and the resulting difficulty persists into algebra and beyond; this is not inevitable but a solvable instructional-design problem. The payoff is substantial. Because early fraction knowledge is among the strongest predictors of later algebra and mathematics achievement, sustained investment in conceptual, magnitude-grounded fraction learning and instruction in the elementary and middle grades is time well spent. When students come to see a fraction as a number with a size—located on the same continuum as the whole numbers they already know—they gain not only competence with fractions but a foundation for the mathematics that follows.

**Key Idea • Fractions reorganize number from counting to magnitude, and a wrong answer tells us less than which understanding a student is still constructing.**

## References

- Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, 113(3), 447–455.
- Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). Academic Press.
- Braithwaite, D. W., Tian, J., & Siegler, R. S. (2018). Do children understand fraction addition? *Developmental Science*, 21(4), e12601.
- Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, 64(3), 293–316.
- Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, 17(3), 283–342.
- Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, 116(1), 45–58.
- Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144). ERIC/SMEAC.

- Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Information Age.
- Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, 38, 201–221.
- National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. U.S. Department of Education.
- Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40(1), 27–52.
- Paik, J. H., & Mix, K. S. (2003). U.S. and Korean children's comprehension of fraction names: A reexamination of cross-national differences. *Child Development*, 74(1), 144–164.
- Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, 23(7), 691–697.
- Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, 17(1), 13–19.
- Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, 62(4), 273–296.
- Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359–371.
- Simon, M. A., Placa, N., Avitur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *The Journal of Mathematical Behavior*, 52, 122–133.
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer.
- Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Kluwer.

## Social Media

**How can a student rename  $\frac{3}{4}$  as  $\frac{6}{8}$ , and still not know whether  $\frac{3}{4}$  is greater than  $\frac{1}{2}$ ?**

Because renaming a fraction is not the same as understanding its size. For years, students read a numerator and a denominator as two separate whole numbers—long before they come to see a fraction as a single quantity with a place on the number line, so they judge  $\frac{1}{8}$  to be greater than  $\frac{1}{3}$  and add  $\frac{1}{2}$  and  $\frac{1}{3}$  to get  $\frac{2}{5}$ . Our new DMT Insight explains the developmental understandings beneath fraction concepts and operations, why they are among the strongest predictors of later achievement, and what a diagnostic built to measure them should look for.

**Inside this Insight, you will learn:**

- Why whole-number reasoning quietly carries over into fractions, and how to surface it
- How a fraction is built by partitioning one into equal units and iterating a unit fraction
- Why the number line represents a fraction as a magnitude and predicts later achievement
- Which key constructs and predictors a fraction Math Learning Profile should be built upon



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