

DMT INSIGHTS

Issue 54 · 2026 · by Jonathan Brendefur, PhD and Associates



MULTIPLICATION AND DIVISION: THE UNDERSTANDINGS THAT BUILD MULTIPLICATIVE THINKING

When the Answer Is Right but the Thinking Isn't

Ask a class of fourth graders what six times eight is, and most will answer forty-eight. Ask those same students why multiplication works, to model it in more than one way, to connect it to division, or to say what happens to the product when both factors double, and the answers diverge. Some describe equal groups, some fall back on repeated addition, and some reason about a relationship that extends well beyond the first problem. The differences go deeper than strategy. They show what the student is thinking about.

This framing defines the construct's scope. Equal groups, arrays, multiplication facts, dividing a total into equal units, and the beginnings of multiplicative comparison, reasoning about one quantity as so many times another, belong to Multiplication and Division. The more developed ratio, rate, and proportional reasoning that grow out of scaling are developed in Scaling and Comparing. This construct lays the foundation for those that draw on it: treating a collection as a unit and coordinating the number of units, the size of each unit, and the total (Verghnaud, 1988; Steffe, 1992).

The change is in the object of a student's reasoning. Additive thinking coordinates individual quantities that are joined, separated, or compared. Multiplication asks the student to treat a collection as a single mathematical object and hold two quantities together while preserving their relationship (Steffe, 1992; Simon & Blume, 1994). What follows concerns the models and language that make that relationship clear enough to reason about.

Key Idea • Getting a product right and understanding the relationship it stands for are different abilities. Multiplication and division mark the point at which students begin to reason about relationships among quantities rather than single quantities.

CLASSROOM SNAPSHOT

Teacher: "Six bags, four apples in each. How many apples in all?"

Student A: "Four, eight, twelve, sixteen, twenty, twenty-four."

Student B: "Six fours. That is twenty-four."

Teacher: "Now, share those twenty-four apples into six bags. How many in each?"

Student B: “Four again. It’s the same fours, just backward.”

The difference is not the answer. It is whether the student counts the apples one at a time or reasons with a composite unit — one four — that can be iterated, shared back out, and compared. Multiplication and division form a single relationship, read in two directions.

The Shift from Additive to Multiplicative Reasoning

Moving into multiplication reorganizes how a student reasons, and that is more than picking up a new procedure. During additive reasoning, students count objects, combine collections, and compare amounts. Multiplication asks them to treat a collection itself as a unit. Four threes (four groups of three) stop being twelve separate objects and become four composite units, each holding three units of one, and those composite units can then be composed, iterated, partitioned, and compared (Steffe, 1992). Bruner (1966) described understanding as developing through enactive, iconic, and symbolic representations that build on one another, which is why bar models, number lines, area models, and ratio tables stay useful long after symbols are introduced.

Division belongs to the same shift. Sharing twenty-four into six equal units (groups) asks how large each unit is; measuring how many fours fit in twenty-four asks how many iterations there are. These are the two meanings of division: sharing fixes the iterator and finds the unit, while measuring fixes the unit and finds the iterator. Both ask the student to reason about the unit rather than to count what is left. When multiplication and division are taught as one relationship, division ceases to be a separate procedure and becomes the same reasoning in reverse.

Why repeated addition is not enough

Repeated addition is a reasonable gateway into equal groups, but it does not, by itself, build the composite unit. A student who reaches forty-eight by adding six eights is still working on ones. The developmental work involves shifting from counting objects to reasoning about the unit as a single item that can be operated on, and from one quantity to the relationship between two (Simon & Blume, 1994; Carpenter, Fennema, & Franke, 1996). Instruction that stops at repeated addition can leave that shift unmade, even as answers come out right.

Key Idea • The move into multiplication changes what the student reasons about, from single quantities to collections treated as units, and to the relationship between two quantities held together at once.

The Understandings That Build Multiplicative Thinking

Research across mathematics education and cognitive psychology points to a small set of understandings that reorganize how students reason and that predict later success. The diagnostic treats them as a developmental progression rather than a checklist. Each one reorganizes what the student attends to, and each keeps working as the next develops, so a student still reasons with composite units while learning to divide and still coordinates quantities while reasoning about proportion.

One question runs through all four: what is the student reasoning about? A student may count objects one at a time, reason with a collection as a unit, coordinate the number of units and the unit size with the total, see a relationship

that holds across representations, or reason about one quantity in relation to another. The four constructs below name those understandings, in the order they tend to develop.

Unit & Iterating. The foundational reorganization is treating a group as a unit while still understanding what composes it, seeing twelve as four units of three, and then iterating the unit: three, six, nine, twelve. A student who can compose, decompose, partition, and iterate equal-sized units rather than count by ones has begun to unitize (Steffe, 1992; Simon & Blume, 1994). Division draws on the same unit: sharing twenty-four into units of six, or finding how many sixes fit in twenty-four, both depend on holding the unit as one item. An array keeps the unit visible while the ones show inside it. Across grades 3 to 6, the unit grows, from a collection of objects seen as one, to a unit of ten held while multiplying, and later to a rate treated as a single unit that can be scaled (Lamon, 2007). Items that reveal it ask what the unit is, how many iterations there are, and whether the student can compose, partition, or iterate the unit rather than counting up from one.

Coordinating Two Quantities. Multiplication coordinates the number of units, the size of each unit, and the total, maintaining the relationship among them. A student who attends only to the five, or only to the eight, in five times eight has not yet coordinated them (Simon & Blume, 1994; Vergnaud, 1988). Given any two of the three, the number of units, the unit size, or the total, the student determines the third; twenty-four shared into six equal units leaves four in each. Division is this same coordination read in reverse. Coordinating these quantities is what sets multiplicative reasoning apart from additive reasoning. Items that reveal it ask what the five and the eight stand for, and what happens to the total if the number of units doubles.

Recognizing Structure. Attention then shifts from the quantities to the relationship that holds across representations and equations. A student who sees that six times eight, forty-eight divided by eight, forty-eight divided by six, an array of forty-eight, and “six times as many” are one relationship seen differently is reasoning about structure rather than recalling separate facts (Vergnaud, 1994; Hiebert & Carpenter, 1992). Students decompose a factor, use the distributive property, and recompose the partial products, reasoning from a known relationship to an unknown one. People who know a domain deeply organize what they know around relationships like these rather than around surface features, which lets them recognize a familiar structure in an unfamiliar problem (Chi, Feltovich, & Glaser, 1981). Items that reveal it ask how two problems are alike, which representation shows the same relationship, and why two strategies give the same answer.

Comparison & Scaling. Students begin to reason about one quantity in relation to another: “times as many,” “times as large,” or an unknown comparison factor. The key distinction is between additive comparison, which asks how many more, and multiplicative comparison, which asks how many times as many; a student who says Leo has eight more when Leo has three times as many has not yet made the shift (Confrey, 1994). Students also begin to reason about simple multiplicative change, how one quantity grows as another does. At this grade, the work stays with whole-number comparison and simple scaling; the more developed ratio, rate, and proportional reasoning it prepares for comes later, in Scaling and Comparing. Items that reveal it ask how many times as many one quantity is than another, and whether a comparison is additive or multiplicative.

Key Idea • Four constructs build multiplicative thinking: Unit & Iterating, Coordinating Two Quantities, Recognizing Structure, and Comparison & Scaling. Each reorganizes what the student reasons about, and each builds on the next.

Where Multiplicative Thinking Leads

Beyond these four, students increasingly recognize and use the same multiplicative structure in new places: equivalent ratios, similar figures, constant rates, linear functions, and the relational reasoning algebra asks for. This generalization is not a fifth construct measured here. It is the outcome of increasingly connected multiplicative reasoning, and it is what prepares students for algebra, which depends on identifying and extending relationships rather than running isolated calculations (Hiebert & Carpenter, 1992; Anderson, 1983; Bransford, Brown, & Cocking, 2000).

Cognitive Psychology Foundations: Structure, Load, and Transfer

Several cognitive psychology findings explain why structured representations help. Cognitive Load Theory (Sweller, 1988) holds that working memory is limited and that learning suffers when additional demands exceed the capacity a student needs for reasoning. Coordinating two quantities and their relationship is demanding, and a bar model, number line, area model, or ratio table provides part of that coordination on the page, freeing capacity for the relationship itself. Dual Coding Theory (Paivio, 1990) adds that information shown both as an iconic model and in symbols is understood and remembered more readily than information held in one form, which is why pairing a model with the symbols strengthens a student's grasp of the same relationship.

The deeper aim is transfer. Students carry what they have learned into new situations when they have grasped the underlying structure rather than the surface features of the problems they first met (Bransford, Brown, & Cocking, 2000; Chi, Feltovich, & Glaser, 1981). Part of the difficulty is that students reach for additive strategies because those form earlier and get practiced for years, so multiplicative reasoning means holding the additive habit back long enough to see the relationship. The recurring errors here are, in that light, moments where structure is missed: treating multiplication as repeated addition and nothing more, reading five times eight as two loose numbers, and assuming that multiplication always makes a quantity larger. Each is best met by turning attention to the multiplicative structure the surface hides, and by keeping the models in play rather than dropping them once symbols appear (Bruner, 1966; Hiebert & Carpenter, 1992).

Key Idea • Structured representations encode the multiplicative relationship so students can reason over it, and the payoff is transfer, which comes from grasping structure rather than surface features.

From Diagnosis to a Math Learning Profile

Read together, these understandings change what assessment is for. Instead of recording which multiplication facts or problem types a student has mastered, a teacher can note which understandings the student has built and which are still forming: whether the student is treating a unit and iterating it, coordinating quantities, recognizing structure, or comparing and scaling. That reading is the basis of a Math Learning Profile, a picture of where a student sits in the progression rather than a single score. It tells a teacher which understanding to build next, and it keeps curriculum, instruction, and intervention pointed at the same target. The representations do double duty here, since the same bar model or ratio table that reveals a gap also serves as the tool for closing it.

Key Idea • A Math Learning Profile names the understandings a student has built and which are still forming, turning a diagnosis into a next instructional move rather than a score.

Conclusion: A Bridge to Proportion and Algebra

Multiplication and division are the earliest sustained chances for students to reason about relationships among quantities. A student who treats a group as a unit and iterates it, coordinates the number of units, the unit size, and the total, recognizes the structure that multiplication and division share, and reasons about how one quantity compares to, and scales with, another is ready for the ratio, proportion, and relational reasoning that algebra asks for. A student who stays with counting and repeated addition meets the same wall again and again as the numbers and the ideas grow. The diagnostic measures not fact fluency but these reorganizations, which research identifies as understandings that predict long-term mathematical success (National Research Council, 2001).

Key Idea • Multiplication and division mark the point at which students begin to reason about relationships among quantities. The understandings they build here, Unit & Iterating, Coordinating Two Quantities, Recognizing Structure, and Comparison & Scaling, are the foundation for proportion and algebra.

References

- Anderson, J. R. (1983). *The architecture of cognition*. Harvard University Press.
- Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How people learn: Brain, mind, experience, and school*. National Academy Press.
- Bruner, J. S. (1966). *Toward a theory of instruction*. Harvard University Press.
- Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3–20.
- Chi, M. T. H., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, 5(2), 121–152.
- Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291–330). State University of New York Press.
- Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). Macmillan.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Information Age.
- National Research Council. (2001). *Adding it up: Helping children learn mathematics* (J. Kilpatrick, J. Swafford, & B. Findell, Eds.). National Academy Press.
- Paivio, A. (1990). *Mental representations: A dual coding approach*. Oxford University Press.
- Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, 25(5), 472–494.
- Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, 4(3), 259–309.
- Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, 12(2), 257–285.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). National Council of Teachers of Mathematics.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–59). State University of New York Press.

Social Media

Ask a class what six times eight is, and most will say forty-eight. Ask why multiplication works, and the answers scatter.

A right answer and an understanding of the relationship behind it are different things. Multiplication and division mark the point at which students begin to reason about relationships among quantities rather than single quantities, and the understandings they build here predict success with proportion and algebra. Our new DMT Insight lays out the four that matter, shows how division is the other side of the same reasoning, and gives you questions to assess each one.

Inside this Insight, you will learn:

- Why a correct product can still hide additive, count-by-ones thinking
- How treating a collection as a unit and coordinating two quantities reorganize a student’s reasoning
- Why sharing and measuring are the two faces of division, not a separate topic
- What bar models, number lines, area models, and ratio tables make visible
- Why these understandings are the foundation for proportion and algebra
- That you can try an MLP diagnostic for grades 3 to 6 at mathsuccess.io



 | mathsuccess.io |  mathsuccess.dmtinstitute.com

CONNECT WITH DMTI

 [@dmtinstitute](https://www.facebook.com/dmtinstitute)  [DMT Institute](https://www.linkedin.com/company/dmt-institute)  [@dmtinstitute](https://www.youtube.com/dmtinstitute)  [@dmtinstitute](https://www.instagram.com/dmtinstitute)  [@dmtinstitute](https://twitter.com/dmtinstitute)  [@math.success](https://www.tiktok.com/@math.success)  [@DMTInstitute.bsky.social](https://bsky.app/profile/dmtinstitute.bsky.social)