

DMT INSIGHTS

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BEYOND THE ALGORITHM: BUILDING THE REASONING BEHIND ADDITION AND SUBTRACTION

When Correct Answers Hide Weak Understanding

A student can carry out the steps for $503 - 297$, write an answer, and never notice whether it could be right. Ask a different question and the gap appears: how far is it from 297 to 503? A student who reasons sees that 297 is almost 300, that 300 to 503 is a little more than 200, and that the difference must be close to 206. Carrying out a procedure and reasoning about the quantities it operates on are different abilities, and the Addition & Subtraction Math Learning Profile (MLP) is designed to show the difference. Across grades K–6, addition and subtraction are less about the steps of an algorithm than about whether a student can compose, partition, and compare quantities and reason about the relationships among them.

Two questions reveal where a student stands. Does the student count on and back through the quantities, or move digits in columns? Moreover, does the student see 503 and 297 as quantities with a size and a distance between them, or as strings of digits to be processed? A student who sees only digits can be accurate for years and still have no way to catch an unreasonable answer or to choose an efficient path.

From a cognitive psychology perspective, much of the difficulty is one of coordination. Holding a quantity, partitioning it, and tracking the units while operating on them taxes working memory, and the load is eased when the quantity is carried in a structured form a student can reason over, such as a bar model or a number line (Hiebert & Carpenter, 1992).

Key Idea • Carrying out an algorithm and reasoning about the quantities it operates on are different abilities. Students get good at addition and subtraction when they stop seeing only digits and start reasoning about quantities, units, structure, and the approximate size of a result.

CLASSROOM SNAPSHOT

Teacher: "Solve $503 - 297$."

Student A: "Three minus seven you cannot, and there is a zero, so I borrowed across it and got 314."

Student B: "297 is close to 300. From 297 up to 300 is 3, and from 300 up to 503 is 203, so the distance is 206."

Teacher: “How do you know 314 cannot be right?”

Student B: “Because 297 and 503 are only about 200 apart, so the answer has to be near 200, not over 300.”

The difference is not the algorithm. It is whether the student is counting the distance between two quantities or moving digits in columns.

From Counting to Reasoning

The transition that defines progress is the move from carrying out procedures to reasoning about quantities, and it begins in the earliest grades. A first grader counting $8 + 9$ by ones may land on 16 instead of 17, having lost a count along the way; a student reasoning with quantity sees $8 + 9$ as one less than $9 + 9$, or completes a ten as $8 + 2 + 7$. The same shift takes a multidigit form later: a student who reasons sees $503 - 297$ as a comparison of two nearby quantities, while a student who only executes sees a column of digits. Both can reach an answer, but only one is positioned for what comes next (Carpenter et al., 1998).

Why students stay with the algorithm

The standard algorithm is available early, has worked for years, and asks nothing about quantity. A student can regroup correctly across three columns, or write $456 + 135$ as 5811 by recording each column without composing a new unit, and in neither case has the student formed a sense of how large the numbers are. The trouble surfaces when the procedure breaks down: in subtraction across zeros, in the common error of taking the smaller digit from the larger in each column, and in answers that are far off without the student noticing. These are not careless slips. They are the predictable result of operating on digits rather than quantities, and each one points back to a missing understanding rather than a missing step (Baroody, 2006).

Carpenter and colleagues followed children across the elementary years and found that those who invented and used flexible strategies grounded in the meaning of the quantities understood multidigit addition and subtraction more deeply, and made fewer systematic errors, than those who relied on the standard algorithm alone (Carpenter et al., 1998). The aim of instruction is therefore not to replace the algorithm but to ensure it rests on reasoning a student can fall back on when the procedure alone fails.

Key Idea • The central transition across the grades is from executing a procedure to reasoning about quantities. Recurring errors in addition and subtraction suggest a student is operating on digits rather than on the quantities the digits represent.

Four Connected Constructs

The Addition & Subtraction MLP is organized around four connected constructs. Together they describe what students need to reason about beyond carrying out a procedure: locating and sizing quantities, composing and renaming base-ten units, using number structure and properties, and estimating to anticipate or judge a result. Each construct names a key developmental understanding, an idea a student must build for addition and subtraction to become meaningful and fluent (Simon, 2006). These four connected constructs are measured by the diagnostic profile, and each item is designed to show how a student reasons, not only whether the student can solve today’s problem.

Quantity & Magnitude. Students reason about how large a quantity is, where it is located relative to benchmark quantities, and the distance between quantities. They compare quantities and reason about how far one quantity is from another or from a benchmark such as the next ten, hundred, or thousand. A student with this understanding expects $503 - 297$ to be close to 206, because 297 is 3 from 300 and 300 is 203 from 503. Students with more accurate representations of numerical magnitude also tend to learn arithmetic more readily (Booth & Siegler, 2008). This construct includes magnitude, benchmark navigation, and comparison as distance.

Composing & Renaming Units. Students compose and decompose complete units of ten, a hundred, and a thousand, and rename a total in equivalent forms. The number 307 can be named as 3 units of hundred and 7 units of one, as 2 units of hundred, 10 units of ten, and 7 units of one, as 30 units of ten and 7 units of one, or as 307 units of one. In each form, the quantity remains the same while its unit structure changes, and each form is useful for a different computation. Regrouping renames the units; it does not borrow them. By the middle grades, the same idea extends to tenths and hundredths, and it carries forward into place value, multiplication, fractions, and decimals.

Structure & Properties. Students use the structure of the quantities and the properties of addition and subtraction to reason. This includes derived facts, compensation, making friendly numbers, decomposition, and the relationship between addition and subtraction. A student who uses structure solves $27 + 38$ by moving 3 across ($27 + 3 + 35 = 30 + 35$), solves $47 - 38$ by taking away 37 and then one more ($47 - 37 - 1 = 10 - 1$), and reasons that $64 - 28$ and $66 - 30$ are equal because both numbers shifted by the same amount. The same reasoning links $46 + 18 = 64$ to $64 - 18 = 46$ and $64 - 46 = 18$. Equality and relational thinking are not a separate construct. They support this one: a student who reads an equation as a statement that both sides name the same quantity can use that balance to make a problem easier (Carpenter et al., 2003). Strategic flexibility is also located here. Choosing a path that fits the quantities, instead of using one procedure for every problem, is evidence that the student sees the structure, and the next section takes it up in detail.

Estimation. Students judge the magnitude of a total or difference before or instead of computing it exactly. They anchor quantities to useful benchmarks, anticipate an approximate result, and use that estimate to judge whether an answer is reasonable. For $503 - 297$, a student who anchors 297 to 300 and 503 to 500 anticipates a result of about 200 and knows that 314 cannot be correct. Estimation is how a student catches an unreasonable answer, and it depends on the other three constructs: a student estimates well by locating quantities, thinking in units, and using structure.

Each construct develops along a trajectory rather than appearing all at once, and each is surfaced most clearly by probing it three ways: recognition, whether the student identifies the relevant quantity or relationship; application, whether the student uses it to solve; and error analysis, whether the student can find and explain another's mistake. Read together, these locate where a student is on the trajectory and which misconception, not merely which answer, is present.

The four constructs interact. A student may use a benchmark because of Quantity & Magnitude, rename a hundred as ten tens through Composing & Renaming Units, use compensation through Structure & Properties, and then use Estimation to judge whether the result is reasonable. Read together, the four constructs provide an instructional profile rather than a single score.

Key Idea • Four connected constructs organize the Addition & Subtraction MLP: Quantity & Magnitude, Composing & Renaming Units, Structure & Properties, and Estimation. The throughline: locate and size the quantity → rename its units → use the structure → estimate to check.

Strategies for Flexible Fluency

Fluency in addition and subtraction is not the speed of recalling facts. It is the flexible, appropriate, and efficient use of strategies grounded in the meaning of the quantities (Baroody, 2006). Four strategies carry most of that work across the grades: using doubles, making and using benchmarks, decomposing by place value, and compensating. All four belong primarily to Structure & Properties, and each also draws on the other constructs. The key understanding is not any one strategy but the habit of choosing among them so the path fits the numbers.

No single strategy is best for every problem, which is why fluency means having more than one. For $17 + 18$, a student can double ($17 + 17 + 1$), use a benchmark to make 20 ($17 + 3 + 15$), or compensate ($17 + 20 - 2$), while decomposing by place value is less efficient here. For $12 + 13$, decomposing by place value works cleanly, while doubling or compensating only adds steps. A student who reaches for the same procedure every time has fluency with one method; a student who reads the numbers first and picks a path has fluency with the operation. The same habit shows in recognizing efficiency: for $297 + 301$, a student who reads the structure adds $300 + 301$ and removes 3, or $297 + 300$ and adds 1, where the standard algorithm is slower and more error-prone, and recognizing that is itself the skill.

Doubles and near-doubles. A student who knows $6 + 6 = 12$ can reason that $6 + 7$ is one more, and that $17 + 18$ is $17 + 17 + 1$. Doubles are derived facts within Structure & Properties: the student works from a known fact rather than recounting (Baroody, 2006).

Benchmarks. Benchmarks are friendly quantities, tens and hundreds most of all, that a student moves toward to simplify a computation. To solve $17 + 18$, a student can make a benchmark of 20: $17 + 3 + 15$. Benchmarks draw on Quantity & Magnitude to locate the friendly quantity and on Structure & Properties to use it, reducing the number of steps a student must hold at once.

Place-value decomposition. A student partitions each number by its units and operates on like units: $12 + 13$ becomes $10 + 10$ and $2 + 3$, then $20 + 5$. This strategy makes Composing & Renaming Units visible, and it is the reasoning the standard algorithm compresses into steps.

Compensation. A student adjusts a number to a friendlier one and corrects for the change: $17 + 18 = 17 + 20 - 2$, or $49 + 27 = 50 + 27 - 1 = 76$. Compensation is a clear case of Structure & Properties, since it depends on seeing that a change to one quantity can be undone elsewhere while the total is preserved (Carpenter et al., 2003).

Key Idea • Strategic flexibility is located primarily within Structure & Properties. It is choosing an efficient path to fit the numbers, not running one procedure for all. Doubles, benchmarks, place-value decomposition, and compensation are the repertoire, and a strategy a student cannot use points back to the construct still forming.

Equality Within Structure & Properties

One part of Structure & Properties deserves particular attention because it is a common obstacle: how students read the equal sign. Many students read the equal sign as an instruction to compute, expecting an answer to follow, rather than as a statement that two quantities are the same. Asked to complete $8 + 4 = \square + 5$, such a student writes 12, treating everything after the equal sign as the place where the answer goes (Carpenter et al., 2003).

A student who understands equality as a relationship can reason that the box must hold 7, because both sides must name the same quantity, and can judge that $64 - 28$ and $66 - 30$ are equal without computing either. This reading of the equal sign supports later work in algebra, where every equation depends on it. Encountering equations in varied forms, true-or-false statements, and open sentences, supported by a balanced bar model, builds the relational reading directly. The same equation can appear as $\square + 38 = 72$, as $38 + \square = 72$, or as $72 = 38 + \square$, and a student who has met only one arrangement often stalls on the others, though the mathematics is identical. Meeting the equal sign in varied positions is part of building the relational reading.

Key Idea • Reading the equal sign as a relationship between two equal quantities, rather than as a signal that an answer follows, is supporting mathematics for Structure & Properties and for the algebra that follows.

How Structure Reduces the Load and Supports Transfer

Reasoning strategies and structured representations help because they reduce what a student must hold in mind at once. Counting $8 + 7$ by ones holds seven separate steps in mind; completing a ten holds two. Compensation and benchmark strategies are efficient precisely because they reduce the number of things a student must track at once, and a bar model offloads part of that tracking onto an external structure. Working memory is limited, and learning suffers when routine demands consume the capacity for reasoning (Sweller, 1988).

Pairing a bar model or a number line with the symbolic computation strengthens a student's grasp of the same relationship, because information shown both visually and symbolically is understood and retained more readily than information held in one form (Paivio, 1990).

Transfer matters as well. Students apply what they have learned to new situations when they have grasped the underlying structure rather than the surface features of the problems they first met (Bransford, Brown, & Cocking, 2000). In this light, recurring errors in addition and subtraction reflect failures to attend to structure: taking the smaller digit from the larger in each column, losing a unit of hundred across a zero, and reading the equal sign as a command. Each is best addressed by directing attention to the quantity structure the digits obscure (Hiebert & Carpenter, 1992).

Key Idea • Reasoning strategies and structured representations reduce the load of operating on quantities, and they support the transfer that comes from grasping structure rather than surface features.

Conclusion: A Foundation for Place Value, Fractions, and Algebra

Addition and subtraction are often treated as basic computational skills, but they are better understood as evidence of how a student reasons about quantity and relationship. The four constructs make that reasoning visible. Quantity &

Magnitude shows whether the student can locate, compare, and reason about distance. Composing & Renaming Units shows whether the student can compose, decompose, and rename base-ten units while preserving the quantity. Structure & Properties shows whether the student can use number structure, compensation, derived facts, and the relationship between addition and subtraction. Estimation shows whether the student can anticipate or judge the magnitude of a result. A student who can do all four is prepared for the place-value work, fraction reasoning, and algebra that follow across grades K–6 and beyond. A student who carries only the algorithm forward meets the same difficulty repeatedly as the mathematics grows more demanding.

Key Idea • Locate and size the quantity, rename its units, use the structure, and estimate to check. The same four constructs continue through place value, fractions, and algebra.

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Social Media

How can a student regroup correctly across three columns and still not see that the answer is far too large?

Because carrying out an algorithm is not the same as reasoning about the quantities it operates on, many students move digits without ever forming a sense of how large the numbers are or how they relate. Our new DMT Insight explains the four connected constructs behind addition and subtraction across grades K–6: Quantity & Magnitude, Composing & Renaming Units, Structure & Properties, and Estimation, and how to assess and teach them so that students stop seeing only digits and start reasoning about quantities, units, structure, and the size of a result.

Inside this Insight, you will learn:

- Why the recurring errors of addition and subtraction point to missing understandings, not missing steps
- The four connected constructs the diagnostic profile measures
- How doubles, benchmarks, place value, and compensation turn reasoning into flexible fluency
- How the four constructs continue through place value, fractions, and algebra

