DMT INSIGHTS

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Math Success by DMTI

SCALING AND COMPARING: THE DEVELOPMENTAL UNDERSTANDINGS

Introduction: A Gateway to Proportional Reasoning and a Genuine Conceptual Challenge

Few transitions in the Grades 3–8 curriculum carry as much long-term weight, or cause as much difficulty, as the move from additive to multiplicative reasoning. Scaling and comparing involve students reasoning about how two quantities change together, how many times as much one is as another, and what happens to a shape, a recipe, or a rate when both quantities grow or shrink by the same factor. It is also when many students' progress stalls, because the additive strategies they used for years begin to fail. Multiplicative and proportional reasoning builds on multiplicative reasoning. It underlies fractions, percent, similarity, slope, and linear functions, and reviews of the research describe it as one of the most protracted and important developments in school mathematics (Tourniaire & Pulos, 1985; Lamon, 2007).

From a mathematics perspective, scaling and comparing is not a single idea but a set of related ones, including multiplicative comparison, scaling, composite units, equivalent ratios, and rate, that students must gradually coordinate (Vergnaud, 1988; Lamon, 2007; Lobato & Ellis, 2010). From a cognitive perspective, learning to reason multiplicatively requires a reorganization of what comparison means: students must move from asking how many more to asking how many times as much, and treat the factor that links two quantities as the object of reasoning rather than the difference between them (Thompson & Saldanha, 2003). Understanding both of these demands, the mathematical structure and the cognitive reorganization, is essential for learning and teaching this material well. These reorganizations are what Simon (2006) termed key developmental understandings, conceptual advances that make later concepts and procedures learnable rather than merely memorable.

> **Key Idea • Multiplicative and scaling reasoning is among the strongest foundations for later mathematics, yet it demands a reorganization of what comparison means, from how many more to how many times as much, which is exactly why scaling and comparing is both consequential and difficult.**

What Is Multiplicative Comparison? The Structure of Scaling and Comparing

A central insight from decades of research is that a multiplicative situation can be read in several related ways. Building on Vergnaud's (1988) analysis of multiplicative structures, researchers describe scaling and comparing as comprising several interrelated subconstructs: multiplicative comparison (how many times as much one quantity is than another), scaling (growing or shrinking a quantity by a factor while its structure stays the same), composite units (treating a ratio such as 3 cups to 2 batches as a single unit that can be iterated and partitioned), equivalent ratios (two ratios related by the same factor), and rate (a relationship between two different units, such as dollars per pound). Competent reasoning requires students to recognize these meanings, distinguish among them, and move flexibly among them (Lamon, 2007; Lobato & Ellis, 2010).

Instruction has traditionally overemphasized procedures, cross-multiplication in particular, at the expense of the underlying reasoning (Lamon, 2007). While a procedure provides a reliable answer, it is a limited foundation. It does not explain why two ratios are equivalent; it hides the factor that links the quantities, and it gives students no way to judge whether an answer is reasonable. Empirical and theoretical work identifies the composed unit and the multiplicative comparison as the ideas that most support equivalence and proportional reasoning (Lobato & Ellis, 2010; Thompson & Saldanha, 2003). A robust curriculum, therefore, deliberately develops all subconstructs, with particular attention to the factor that relates two quantities rather than to the steps of a rule.

Key Idea • Scaling and comparing is not one idea but several: multiplicative comparison, scaling, composite units, equivalent ratios, and rate, and instruction should develop all of them, with particular attention to the multiplicative factor.

The Additive Bias: Reorganizing Comparison

Perhaps the single most documented obstacle in this domain is the tendency to treat multiplicative situations additively (Tourniaire & Pulos, 1985; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). Because students spend their early years reasoning about how many more and how many fewer, they build strong, well-practiced intuitions: comparison means finding a difference, and a quantity grows by adding to it. These intuitions succeed for years, and they are often the correct reading. They fail when the situation is multiplicative, and the additive habit continues.

The consequences are familiar to every upper-elementary teacher. Asked to enlarge a 4-by-6 rectangle so its width becomes 6, students add 2 to the height and report 8, keeping the difference while changing the shape. Asked to double a recipe that calls for 3 cups of flour for 2 batches, they add the same amount to each batch and change the recipe. They read a ratio of 3 to 4 as "a 3 and a 4," two separate numbers, and extend a proportion by adding to each. Van Dooren and colleagues (2005) documented how readily students apply additive and proportional methods where they do not belong, a pattern that persists across ages and problem types. From this perspective, the goal of assessment and instruction together is not merely to add a new rule but to help students reorganize comparison so that the factor linking two quantities becomes what they track; a diagnostic that surfaces where that reorganization has stalled makes the learning teachable.

Key Idea • The additive bias, reading a multiplicative situation as a difference to be added, is the central obstacle in scaling and comparing; the goal of diagnosis and instruction together is not more rules but a reorganization of comparison so that the multiplicative factor becomes the object of reasoning.

CLASSROOM SNAPSHOT

Teacher: "This recipe uses 3 cups of flour for 2 batches. What would we need for 4 batches?"

Student A: "Add 2 cups and 2 batches. So 5 cups for 4 batches."

Student B: "Double both numbers. 6 cups for 4 batches."

Teacher: "Why?"

Student B: "Because it still has to taste the same."

Both students can compute; they differ in whether they track the factor that relates the quantities or the difference between them, which is the additive bias.

Scaling and the Number Line: From Iteration to Proportion

A mathematical model critical to this development is the number line. On a single line, scaling by a whole number means repeating equal jumps: three jumps of four reach twelve. Locating a fraction is scaling by a factor below one: one-half of the distance from zero to two lands at one. The same line that shows where 12 is relative to 4 also shows where one-half of 2 is, which connects multiplicative comparison with whole numbers to scaling by fractions. Bruner (1966) described learning as a movement from enactive through iconic to symbolic forms, and the number line is at the iconic stage, giving spatial meaning to relationships before they are written symbolically.

The evidence for the number line is strong. A meta-analysis by Schneider and colleagues (2018) found a consistent association between number-line estimation and broader mathematical competence, stronger for fractions than for whole numbers, and a secure sense of numerical magnitude is itself a strong predictor of later achievement (Booth & Siegler, 2008). The double number line extends the same idea to two quantities at once. Developed within Realistic Mathematics Education, it shows two measures scaling together along parallel lines; Freudenthal (1973) and Gravemeijer (1999) described how such a model begins as a model of a specific situation and becomes a model for proportional reasoning in general. Used as a scaling model rather than a counting strip, the number line lets students coordinate two quantities, place a value by reasoning about its magnitude, and apply the same scaling to ratio, proportion, and slope (Lesh, Post, & Behr, 1987).

> **Key Idea • Used as a scaling model rather than a counting strip, the number line connects whole-number iteration to fraction and proportional reasoning within one representation, and number-line placement is among the strongest correlates of later mathematics achievement.**

Developmental Foundations: Composite Units and Coordinating Quantities

The number line makes a multiplicative relationship visible, but students can use it meaningfully only after they have built the understandings beneath it, above all the composite unit. Long before students meet formal ratio notation, they have intuitive resources for multiplicative reasoning, reasoning about equal groups, fair shares, and simple scaling in familiar contexts, and these informal strategies form a productive foundation (Carpenter, Fennema, & Franke, 1996). The developmental studies of proportional reasoning, Noelting's (1980) analysis of children comparing orange-juice mixtures and Karplus, Pulos, and Stage's (1983) study of early adolescents on rate problems, showed that students move through identifiable stages, first comparing within a single quantity and only later coordinating two quantities at once.

Two developmental ideas are central. The first is the composite unit: treating a ratio such as 3 cups to 2 batches as a single unit that can be iterated to reach 6 to 4 and 9 to 6, and partitioned to reach smaller equivalent ratios (Lamon, 2007; Lobato & Ellis, 2010). The second is unitizing: understanding what counts as "one" and recognizing that the unit can be redefined, that a group of objects, or a pair of quantities, can itself be treated as a single unit. Coordinating the composed unit is what allows a student to generate and compare equivalent ratios rather than track four separate numbers. Realistic Mathematics Education emphasizes building these ideas from meaningful contexts of sharing, mixing, and measurement before formalizing them (Freudenthal, 1973; Gravemeijer, 1999). The instructional implication is that multiplicative reasoning should be built from students' own coordinating activity rather than introduced as ready-made symbolic rules.

> **Key Idea • Scaling and comparing should be built from students' own coordinating activity, forming a ratio as one composed unit, then iterating and partitioning it, rather than introduced as ready-made procedures.**

Cognitive Psychology Foundations: Structure, Load, and Transfer

Several findings from cognitive psychology explain why structured representations help. Cognitive Load Theory (Sweller, 1988) holds that working memory is limited and that learning suffers when extra demands use the capacity needed for reasoning. A multiplicative comparison is demanding because it asks a student to coordinate two quantities and a factor at once. A double number line or a ratio table shifts part of that coordination onto an external structure, freeing capacity for the relationship itself, and information represented both visually and symbolically is understood and retained more readily than information held in a single form (Paivio, 1990).

This connects to a broader theme in the psychology of mathematics: the relationship between conceptual knowledge and procedural knowledge. The two are not competing goals; they develop most productively together, and when a procedure such as cross multiplication is learned apart from the multiplicative relationship, students have no way to judge whether an answer is reasonable (Hiebert & Carpenter, 1992). The deeper goal is transfer: students apply what they have learned to new situations when they have grasped the underlying structure rather than the surface features of the problems they first met (Bransford, Brown, & Cocking, 2000). The recurring errors in this domain, which add to the difficulty of preserving a comparison, reading a ratio as two numbers, and assuming that multiplication always makes a quantity larger, are best addressed by directing attention to the multiplicative structure that the surface features hide.

> **Key Idea • Conceptual and procedural knowledge grow together; when a procedure is learned apart from the multiplicative relationship, students cannot tell an unreasonable answer from a correct one, which is why the factor must anchor computation.**

Building the Scaling and Comparing Math Learning Profile: What the Diagnostic Should Measure

The research reviewed here leads to a practical question: if reasoning multiplicatively is this consequential and this fragile, what should a diagnostic actually measure? A Math Learning Profile (MLP) for scaling and comparing is a map of the developmental understandings a student has, and has not yet constructed, built so that a teacher can see where a student's reasoning begins to break down. The literature points to a small set of constructs that predict later proportional reasoning and that together define what such a profile should assess.

Four understandings are primary, because they are the ideas that scaling and comparing rest upon. The first is scaling: whether a student can grow or shrink a quantity by a factor while its structure stays the same, so that a figure keeps its shape and a recipe keeps its taste (Lobato & Ellis, 2010). The second is multiplicative comparison: whether a student asks how many times as much one quantity is as another and tracks that factor rather than the difference between the quantities, the understanding where the additive bias most often intrudes (Thompson & Saldanha, 2003). The third is rate reasoning: whether a student can coordinate two different units, such as dollars per pound, and treat the rate as a single relationship rather than two loose numbers (Lamon, 2007). The fourth is composite units: whether a student can treat a ratio such as 3 cups to 2 batches as one unit that can be iterated and partitioned to generate and compare equivalent ratios (Lamon, 2007; Lobato & Ellis, 2010).

Two further understandings are drawn from related constructs and assumed rather than taught here: multiplicative thinking in equal groups and factors, which grows out of the multiplication facts and is developed in Multiplication and Division (Carpenter, Fennema, & Franke, 1996), and fraction magnitude, the sense that a factor can be greater than, equal to, or less than one, which is developed in Fraction Concepts (Booth & Siegler, 2008). A diagnostic built on these constructs probes each in more than one representation: ratio table, double number line, and drawings of figures that grow and shrink, and surfaces the additive bias directly, because a student who adds where the situation asks them to scale has revealed a specific, addressable misconception rather than a wrong answer. Read across the developmental trajectory, these constructs locate not merely whether a student can solve a proportion today, but which understanding, if strengthened, would change what the student can do next.

> **Key Idea • A scaling and comparing Math Learning Profile is built on the developmental understandings that predict later reasoning, scaling, multiplicative comparison, rate reasoning, and composite units, each probed across representations, where the diagnostic identifies which understanding a student is still constructing.**

Conclusion: A Reorganization from Additive to Multiplicative Reasoning

Scaling and comparing occupy a pivotal place in the Grades 3–8 curriculum because they demand something new: a reorganization of comparison from finding a difference to relating two quantities. From a mathematics perspective, teaching it well means developing the full set of subconstructs: multiplicative comparison, scaling, composite units, equivalent ratios, and rate, with particular attention to the factor that links two quantities. From a cognitive perspective, it means helping students overcome the additive bias and coordinate two quantities at once, so that conceptual understanding of the multiplicative relationship can anchor and give meaning to procedures.

The payoff is substantial. Because multiplicative and proportional reasoning underlies so much of the mathematics that follows, sustained investment in it during the elementary and middle grades is among the higher-leverage decisions a school can make. When students come to ask how many times as much rather than how many more, and to treat that factor as the thing to find and operate on, they gain not only competence with ratios and proportions but a foundation for fractions, percent, similarity, slope, and the algebra ahead.

> **Key Idea • Scaling and comparing is a reorganization of comparison from adding to scaling; investing in multiplicative, factor-grounded learning and instruction in the elementary and middle grades is among the highest-leverage decisions a school can make.**

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Social Media

Two fourth graders. Same recipe question. Both can compute.

Teacher: "This recipe uses 3 cups of flour for 2 batches. What would we need for 4 batches?"

Student A: "Add 2 cups and 2 batches. So 5 cups for 4 batches."

Student B: "Double both numbers. 6 cups for 4 batches."

Teacher: "Why?"

Student B: "Because it still has to taste the same."

Student A is not careless. For years, comparing has meant finding a difference, and that habit is usually correct. It fails the moment the situation is multiplicative, which is why a student who knows 6 x 4 can still enlarge a photo the wrong way.

Our new DMT Insight examines what must change for Student A to become Student B.

- Why the additive bias carries into multiplicative situations, and how to surface it
- How scaling and comparing are built from composite units and coordinating two quantities
- Why the number line, used as a scaling model rather than a counting strip, is among the strongest correlates of later achievement
- Which developmental understandings a scaling and comparing Math Learning Profile should be built upon

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