

DMT INSIGHTS

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PROGRESSIVE FORMALIZATION: FROM MATHEMATICAL RELATIONSHIPS TO FORMAL PROCEDURES

Research Overview

Introduction

Progressive formalization describes how students develop and formalize mathematical relationships through quantities, representations, mathematical language, and procedures. Students may begin with informal strategies, contextual problems, visual models, equations, or familiar procedures. Through instruction, discussion, and practice, students learn to connect these approaches and use mathematical procedures accurately, efficiently, and flexibly.

In DMTI's approach, formalization involves more than moving from concrete objects to abstract symbols. Students identify units and quantities, compose and decompose numbers, examine mathematical structure, and connect these relationships to formal mathematical notation and procedures.

Conceptual understanding and procedural fluency develop together. Teachers connect representations, mathematical explanations, and procedures through explicit instruction, feedback, and practice.

Key Idea • Formalization connects units, quantities, and relationships to formal notation and procedures. Understanding and fluency develop together.

What Is Progressive Formalization?

Progressive formalization involves developing and expressing mathematical ideas in increasingly precise and generalizable ways. Informal strategies, models, mathematical language, and formal procedures can all contribute to this development.

Freudenthal's work on mathematization emphasizes developing mathematical relationships through problem solving and mathematical activity (Freudenthal, 1991; van den Heuvel-Panhuizen & Drijvers, 2014). Bruner's (1966) descriptions of enactive, iconic, and symbolic representations offer another perspective on how students can represent mathematical ideas. These perspectives help explain how students can represent and formalize mathematical ideas. They do not establish a required instructional sequence.

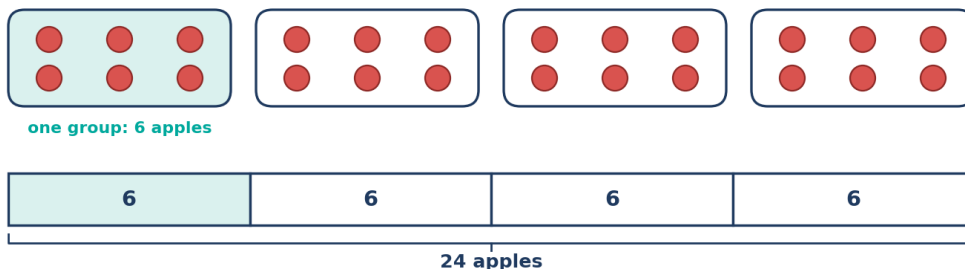
Formalizing Multiplication Through Mathematical Relationships

Multiplication shows how students connect a contextual problem, representations, and a formal procedure. The three ideas below are connected, and students may work with them in different orders.

A. Identify units and quantities. Four baskets contain 6 apples each. How many apples are there altogether? Students identify a group of 6 apples as the unit being iterated. Four such groups compose a total of 24 apples (Figure 1).

$$4 \times 6 = 24$$

Four baskets with 6 apples in each



4 groups of 6: $4 \times 6 = 24$

The unit being iterated: one basket of 6 apples

Figure 1. Four baskets of 6 apples. A group of 6 apples is the unit, iterated four times.

B. Represent and decompose relationships. Students can decompose 14 into 10 and 4, giving 10 groups of 6 and 4 groups of 6. An array, area model, or ratio table can represent this relationship (Figure 2 and Table 1).

$$14 \times 6 = (10 \times 6) + (4 \times 6) = 60 + 24 = 84$$

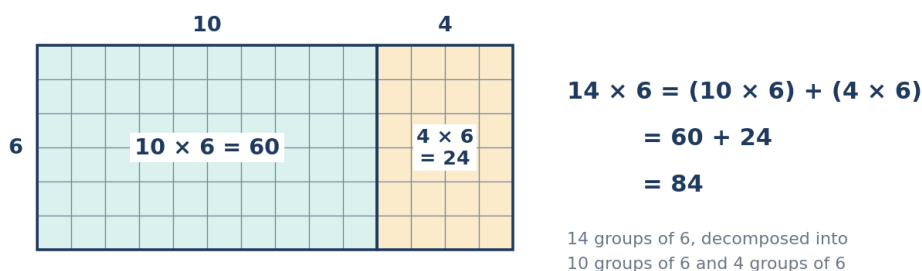


Figure 2. An area model for 14×6 : 14 groups of 6, decomposed into 10 groups of 6 and 4 groups of 6. Each column of 6 squares represents one group of 6.

The important mathematical idea is the distributive property. In the ratio table, each column shows the number of groups and the corresponding number of apples. Combining 10 groups of 6 with 4 groups of 6 gives 14 groups of 6, or 84 apples.

Table 1. A ratio table for groups of 6 apples.

Number of groups	1	2	4	10	14
Number of apples	6	12	24	60	84

C. Connect the relationship to formal procedures. The standard multiplication algorithm uses place value and the distributive property to organize the calculation efficiently. Teachers can connect the algorithm to partial products, the area model, and the ratio table, showing how each represents place value and the distributive property (Figure 3).

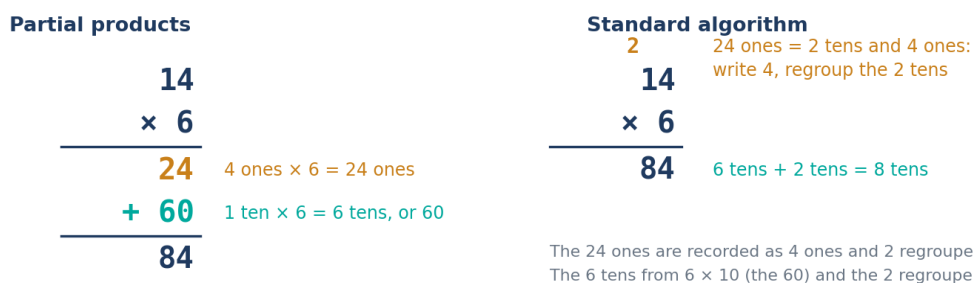


Figure 3. The standard algorithm for 14×6 beside the partial products. Connect the digits and regrouping to the quantities represented by the partial products. The 60 is represented by the 6 tens from 6×10 , which combine with the 2 regrouped tens to make 8 tens.

Students do not need to use every representation before they learn the algorithm. Teachers select representations that make the unit, the decomposition, and the place-value relationships visible to students.

Key Idea • The standard algorithm organizes the same relationship the partial products show: $14 \times 6 = (10 \times 6) + (4 \times 6)$.

Research on Formalization and Mathematical Learning

Research on mathematics learning helps explain why connecting mathematical representations, properties, and procedures matters. Students benefit from connecting mathematical representations to the quantities and relationships they describe. These connections can help make properties such as the distributive property explicit and provide a basis for understanding formal procedures. Izsák (2004) examined how students coordinated units and representations when learning two-digit multiplication, illustrating the importance of attending to the mathematical relationships represented by a model.

Research on conceptual and procedural knowledge suggests that these forms of knowledge develop together (Rittle-Johnson et al., 2001; Rittle-Johnson et al., 2015). Students need opportunities to examine mathematical relationships while also receiving explicit instruction and practice with efficient procedures.

Cognitive load theory also suggests that teachers should consider students' prior knowledge and task demands when selecting representations and explanations (Sweller, 1988). A representation that helps one student may be unnecessary or confusing to another (Kalyuga et al., 2003).

Instructional Considerations

Teachers need to make the mathematical relationships underlying procedures explicit. This includes identifying the units and quantities involved, selecting appropriate representations, and connecting student strategies to mathematical properties and notation.

Student responses show how students interpret a task. Teachers can use this evidence to determine when students need additional explanation, a different representation, or more practice.

Formal procedures should be taught explicitly and practiced. Students also need opportunities to explain how those procedures represent mathematical relationships.

Implications for Curriculum and Assessment

Curriculum should provide opportunities for students to connect contextual problems, mathematical models, equations, and formal procedures. These connections should be intentional but need not follow a fixed instructional sequence.

Assessments should examine students' accuracy and procedural fluency as well as their ability to interpret quantities, represent relationships, and explain their reasoning.

Summary

Progressive formalization involves developing mathematical relationships that students can represent, explain, and use through formal procedures. Understanding the units, quantities, and properties involved helps students make sense of the mathematics underlying those procedures.

Teachers support this development through explicit instruction, mathematical discussion, feedback, and practice. Conceptual understanding and procedural fluency develop together as students learn to apply mathematical relationships accurately, efficiently, and flexibly.

References

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How are the partial products 60 and 24 represented in the standard algorithm for 14×6 ?

Progressive formalization describes how students develop and formalize mathematical relationships through quantities, representations, mathematical language, and procedures. Conceptual understanding and procedural fluency develop together. Inside this DMT Insight, we look at:

- What progressive formalization means, and why it does not require a fixed sequence of representations
- How 4×6 and 14×6 make the unit, decomposition, and the distributive property explicit
- How partial products, area models, and ratio tables connect to the standard algorithm
- What research on conceptual and procedural knowledge and cognitive load suggests for instruction

Read the full Insight at mathsuccess.io/dmtinsights.

