

DMT INSIGHTS

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BEYOND BORROWING: UNDERSTANDING THE STRUCTURE OF SUBTRACTION

How place-value units are composed and decomposed in multi-digit subtraction

Introduction

Multi-digit subtraction is often taught through a sequence of steps commonly described as “borrowing.” Students may learn to cross out digits, rewrite numbers, and follow the standard algorithm without understanding the place-value relationships those procedures represent.

Research on children’s mathematical thinking has examined how students develop multi-digit subtraction strategies, how they understand tens and ones, and why certain errors occur when regrouping. This research offers important insights into the relationship between place-value knowledge, mathematical representations, and written procedures.

This DMT Insight reviews research on the development of multi-digit subtraction strategies, the mathematical structure underlying regrouping, and the connections between conceptual understanding and procedural fluency.

Subtraction and Place-Value Relationships

Subtraction can represent removing a quantity, determining a difference, or finding an unknown quantity. Selter et al. (2012) examine taking away and determining the difference as two important models of subtraction. These relationships influence students’ strategies, but multi-digit subtraction also requires understanding how place-value units represent quantities. DMT Insight 45 examines subtraction situations and their relationships in greater detail. Here, the emphasis is on how students develop place-value understanding and connect that knowledge to subtraction procedures.

Place-Value Knowledge and Multi-Digit Subtraction

Fuson (1992) reviewed research on whole-number addition and subtraction and described how children’s multi-digit calculation depends on understanding numbers as composed of units of different sizes. A written numeral such as 43 records four units of ten and three units of one. Children who treat digits as independent single-digit numbers can follow steps that look correct while losing track of the quantities those digits represent.

Fuson and colleagues (1997) examined how children’s conceptual structures for multi-digit numbers relate to their addition and subtraction methods. Their work shows the importance of understanding tens and ones as units rather than treating written numerals as independent digits. This distinction becomes especially important when a subtraction problem requires regrouping. A quantity such as 43 can be represented as four tens and three ones or three tens and thirteen ones. The representation changes, but the quantity stays the same.

$$43 = 4 \text{ tens} + 3 \text{ ones}$$

$$43 = 3 \text{ tens} + 13 \text{ ones}$$

Teachers’ understanding of this structure shapes how they explain the algorithm. Ma (1999) found that many U.S. teachers described regrouping as “borrowing,” while many Chinese teachers explained it as decomposing a higher-value unit and connected it to composing units in addition. DMTI’s materials describe regrouping in the same terms, as decomposing one unit into ten units of the next smaller place (Brendefur et al., 2021). That language is an instructional framework consistent with this research, not a finding the studies establish.

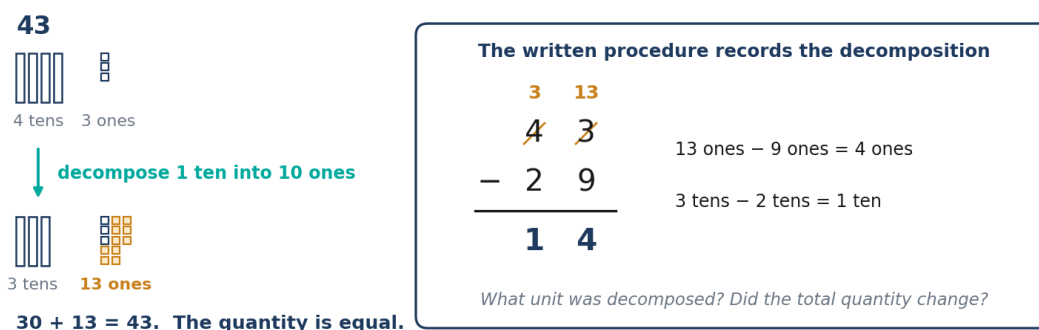


Figure 1. In $43 - 29$, one ten is decomposed into ten ones. The quantity remains 43, and the written procedure records the decomposition.

Key Idea • Regrouping decomposes one unit into ten units of the next smaller place. The representation changes. The quantity does not.

Research on Children’s Multi-Digit Subtraction Strategies

Children use a range of strategies when solving subtraction problems, including direct modeling, counting back, counting up, and reasoning from known number relationships. Research on children’s mathematical thinking shows that these strategies vary with the problem, the numbers, and children’s developing knowledge (Carpenter et al., 1999). Siegler (1996) describes how children use multiple strategies and how their strategy choices change as their knowledge develops.

Carpenter et al. (1998) followed 82 children from first through third grade and interviewed them repeatedly about multi-digit addition and subtraction. About 90 percent of the children used invented strategies at some point. Children who used invented strategies before learning standard algorithms showed stronger knowledge of base-ten number concepts and were more successful at extending their knowledge to new situations than children who learned the algorithms first. Because the children were observed rather than assigned to different instruction, the study describes an association between strategies and understanding, not the effect of a particular teaching sequence.

Fuson et al. (1997) connected children's methods to their conceptual structures for two-digit numbers. Children who think of a number as a sequence of tens and ones tend to count on or back by tens, while children who think of separate tens and ones tend to operate on each place. Beishuizen (1993) found that Dutch second graders who split both numbers into tens and ones made more errors on subtraction problems that required regrouping than students who kept the first number whole and subtracted tens and then ones. For $43 - 29$, splitting produces $40 - 20$ and $3 - 9$, and the second step cannot be completed without regrouping. Jumping from the first number, $43 - 20 = 23$ and $23 - 9 = 14$, avoids that step.

Understanding Regrouping and Written Algorithms

The standard algorithm records regrouping compactly. At each place, it answers the same question: Are there enough units in this place to subtract? If not, one unit of the next greater place is decomposed into ten units of this place. In $402 - 178$, there are not enough ones, and the tens place contains no tens to decompose. One hundred is decomposed into ten tens, and one of those tens is decomposed into ten ones:

$$402 = 4 \text{ hundreds} + 0 \text{ tens} + 2 \text{ ones}$$

$$402 = 3 \text{ hundreds} + 10 \text{ tens} + 2 \text{ ones}$$

$$402 = 3 \text{ hundreds} + 9 \text{ tens} + 12 \text{ ones}$$

The 3 written above the 4 records three remaining hundreds, the 9 above the 0 records nine tens, and the 12 records twelve ones. Each mark corresponds to an equivalent representation of 402. Fuson (1992) describes written multi-digit methods as records of these unit exchanges. When students understand the exchanges, the marks have meaning. When they do not, the notation becomes a set of rules to remember, and problems with zeros in the minuend are a common place for errors (Brown & VanLehn, 1982).

What Research Reveals About Subtraction Errors

Brown and VanLehn (1982) examined systematic errors in subtraction procedures and described how students may construct incorrect rules when they encounter steps they do not know how to complete. Many of the errors they analyzed involve borrowing, particularly from a zero. Such errors can reveal students' procedures, but a single incorrect answer is not enough to show how a student understands the mathematics.

For example, a student who writes $43 - 29 = 26$ may be subtracting the smaller digit from the larger digit in each column ($9 - 3 = 6$ and $4 - 2 = 2$). The same answer could come from a one-time mistake or from a rule the student applies consistently. Distinguishing a mistake from a misconception requires more evidence than one response, such as the student's explanation or work on other problems that require regrouping.

Hiebert and Wearne (1986) studied students' knowledge of decimal numbers and found that many students learned rules for manipulating written symbols without connecting them to the quantities the symbols represent. Their study concerns decimals, but it describes the same broader pattern: procedures carried out without reference to units are more likely to break down in unfamiliar cases.

Representations, Mathematical Structure, and Procedural Understanding

Research on mathematical representations emphasizes that models can help students organize quantities and relationships. In multi-digit subtraction, base-ten representations can make the composition and decomposition of units visible, while number lines can represent differences and relationships among quantities. The instructional significance lies in how students use and connect these representations to their strategies and written procedures.

Realistic Mathematics Education describes models that begin as situation models and become models for mathematical reasoning (Gravemeijer, 1994). Treffers (1991) described a primary mathematics program in which written procedures develop gradually from students' informal methods. Neither framework requires students to complete a fixed sequence from concrete materials to pictures to symbols. Representations and written procedures can develop together as students connect them.

Implications of the Research

Research on multi-digit subtraction suggests that students' understanding of place-value units is closely connected to the strategies and procedures they use. Studies of children's invented strategies, conceptual structures, and systematic errors show why teachers need to understand the mathematics students represent.

This research also supports connecting mathematical representations, student reasoning, and written procedures. Yackel and Cobb (1996) describe how classroom norms for what counts as an acceptable mathematical explanation develop as students explain and compare their methods. Conceptual understanding and procedural fluency should develop together, with opportunities for students to examine relationships and practice increasingly efficient methods.

Key Idea • Understanding place-value units and practicing efficient procedures develop together. Neither has to wait for the other.

Conclusion

Research on multi-digit subtraction shows the importance of students' understanding of place-value units and the relationships represented by their strategies and procedures.

Fuson and colleagues examined how children's understanding of multi-digit numbers influences their calculation methods. Carpenter and colleagues documented the development of children's invented strategies, while Brown and VanLehn identified systematic errors that can occur when students apply written procedures without understanding particular steps.

Together, these studies help explain why regrouping should be understood as composing and decomposing equivalent quantities rather than simply following a sequence of written steps.

The goal is for students to understand the mathematical relationships represented by subtraction procedures and develop accurate, efficient methods they can use across different situations.

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Social Media

Regrouping is not borrowing.

Consider $402 - 178$. No tens to decompose, so a hundred becomes ten tens and a ten becomes ten ones. The algorithm's marks record those exchanges.

In DMT Insight 25, we review research on place-value understanding, children’s multi-digit subtraction strategies, and systematic errors in written procedures.

Read the full Insight at mathsuccess.io/dmtinsights.

How do your students explain the marks they write when they regroup?

