

DMT INSIGHTS

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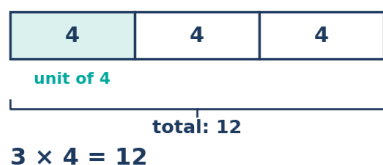
DEVELOPING MULTIPLICATIVE REASONING: UNITS, ITERATION, AND STRUCTURE

Connecting bar models, area models, and multiplication procedures

Introduction

Multiplication involves identifying a unit, determining how many times it is iterated, and understanding the relationship between the unit, the iterations, and the total. For example, 3×4 represents three iterations of the unit 4, resulting in 12 (Figure 1).

Bar model: 3 iterations of the unit of 4



Area model: 3 by 4

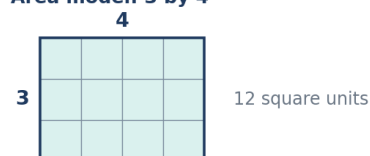


Figure 1. 3×4 as three iterations of the unit of 4 in a bar model, and as a 3-by-4 area model.

Students develop multiplicative reasoning as they examine these relationships through bar models, area models, numerical expressions, and multiplication procedures. Structural language helps students identify the quantities and explain how the units are composed and decomposed.

This Insight examines how units, iteration, mathematical structure, and progressive formalization connect students' understanding of multiplication with increasingly efficient procedures.

Key Idea • Multiplication involves a unit, the number of times it is iterated, and the resulting total. Bar and area models make these quantities visible.

Structural Language

DMTI uses structural language to describe what the numbers in a multiplication expression represent and how the quantities are related.

Term	DMTI explanation
Unit	A quantity treated as one unit that can be iterated or composed with other units
Iterate	Repeat a unit a specified number of times.
Compose	Combine quantities or units to form another quantity.
Decompose	Partition a quantity into parts while preserving its total value.
Total	The resulting quantity represented by the multiplication expression

In 3×4 , the unit of 4 is iterated three times to produce a total of 12. The same quantity can be decomposed as $(2 \times 4) + (1 \times 4)$. In a bar model, the three iterations are partitioned into two iterations and one iteration, and the total remains 12. The unit depends on the mathematical situation. For example, 3×4 can represent three iterations of the unit of 4, while the commutative relationship allows students to examine four iterations of the unit of 3.

Structural language helps teachers and students describe what the numbers represent and how the quantities relate. Terms such as unit, iterate, compose, and decompose should be connected to the representations and procedures students use. Research on children's multiplication shows that their strategies develop from counting single items to operating on composite quantities (Mulligan & Mitchelmore, 1997). This research supports attention to the structure of multiplication; it does not test the effect of particular terms.

Developing Multiplicative Reasoning

Students develop multiplicative reasoning as they learn to identify and operate on composite units (Mulligan & Mitchelmore, 1997). Bar models can represent the number of iterations and the quantity being iterated. Area models organize multiplicative relationships through a rectangle's dimensions.

Teachers can connect these representations to numerical expressions, multiplication properties, and procedures. For example, decomposing 24 into 20 and 4 shows how multiplication distributes across the two quantities. The partial products are then composed to determine the total.

Instruction should connect representations and procedures while responding to students' prior knowledge. Conceptual understanding and procedural fluency develop together through explicit instruction, feedback, and practice (Kilpatrick et al., 2001; Rittle-Johnson et al., 2001).

Bar Models and Area Models

Bar models and area models are the principal representations in this approach. Each makes different relationships visible.

Bar models. Bar models represent iterations of a unit, part-total relationships, and comparisons involving multiplication. In $4 \times 6 = 24$, four iterations of the unit of 6 produce a total of 24. Students can partition the bar to show

$4 \times 6 = (2 \times 6) + (2 \times 6)$, which makes the decomposition of the iterations visible. Similarly, seven iterations of the unit of 8 can be decomposed into five iterations and two iterations (Figure 2).

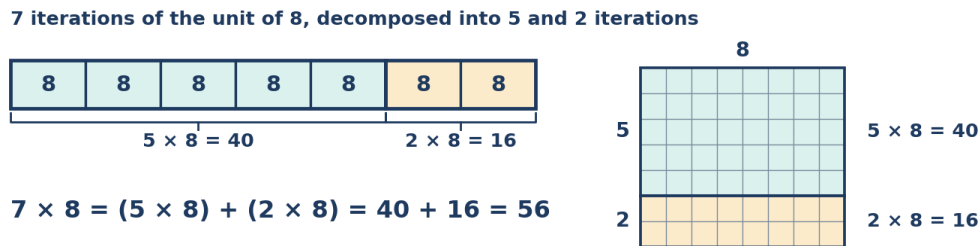
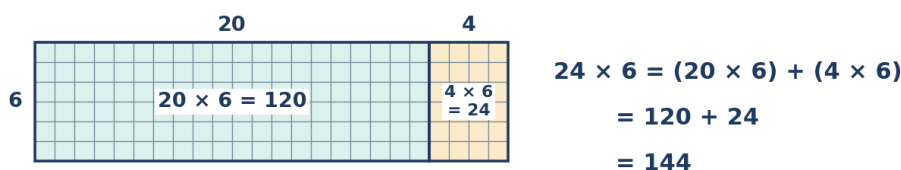


Figure 2. 7×8 decomposed into 5×8 and 2×8 in a bar model and an area model.

Area models. Area models represent multiplication through two linear dimensions and the area they determine. They are particularly useful for examining the distributive property and place-value decomposition. The model for 24×6 can be partitioned into rectangles with dimensions 20×6 and 4×6 . The partial areas, 120 and 24 square units, compose a total area of 144 square units (Figure 3). Izsák (2004) examined how students coordinated units when using area models for two-digit multiplication.



The partial areas compose the total area.

Figure 3. 24×6 decomposed by place value into 20×6 and 4×6 .

Equations, partial products, and algorithms. These represent and calculate the same relationships. A standard algorithm organizes place-value relationships efficiently, and teachers can explicitly connect its steps to the distributive property and the partial products.

Ratio tables can also organize multiplicative relationships and are useful for proportional reasoning. The representations in this Insight are connected through the same quantities and properties; they are not a required instructional sequence.

Progressive Formalization

Progressive formalization describes how students develop increasingly efficient and generalizable mathematical methods while maintaining connections to the quantities and relationships they represent (Gravemeijer, 1994, 1999).

In multiplication, bar models, area models, partial products, and standard algorithms can represent related mathematical structures. Teachers can make these connections explicit through instruction, worked examples, feedback, and practice, developing conceptual understanding and procedural fluency together.

Patterns and Structure

Changing the number of iterations or the unit size changes the total in predictable ways.

$$3 \times 4 = 12 \quad 6 \times 4 = 24 \quad 3 \times 8 = 24$$

Six iterations of the unit of 4 are twice as many iterations as three, so the total doubles. In 3×8 , the unit doubles from 4 to 8 while the number of iterations stays at 3, so the total also doubles.

These examples help students examine the relationship among the unit, the number of iterations, and the total. These relationships also provide a basis for reasoning about scaling and proportional relationships.

Key Idea • Doubling the unit or doubling the number of iterations doubles the total.

Summary

Multiplicative reasoning develops as students learn to identify units, iterate quantities, and examine the relationships represented by multiplication. Bar models, area models, and numerical expressions provide different ways to represent these relationships.

Teachers can use these relationships to connect decomposition, the distributive property, partial products, and standard algorithms. Explicit instruction, feedback, and practice help students develop accurate, efficient, and flexible multiplication procedures.

References

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Social Media

What does 3×4 represent?

Three iterations of the unit of 4 produce a total of 12. A bar model shows the unit being iterated, and an area model represents the same multiplication through the dimensions of a rectangle. This DMT Insight examines:

- How units and iteration describe multiplication
- How bar and area models represent decomposition and the distributive property
- How these relationships connect to partial products and efficient procedures

Read the full Insight at mathsuccess.io/dmtinsights.

When your students multiply, can they name the unit and the number of iterations?

